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The Structure of all Possible Solutions to the Buckley-Leverett Model

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Abstract

We consider a free boundary problem for a one-dimensional system of Buckley-Leverett equations, describing the displacement of oil by a suspension. For this problem we formulated conditions for the strong decay of the discontinuity of the initial oil concentration. We will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity in the initial concentration of oil, when at rest in one half of the domain there is oil, and in the other half of the domain there is a suspension, and these domains are separated by an impenetrable partition. At the initial moment in time, the partition is removed, which initiates the movement of the fluids. A precise analysis of the unique solution to the corresponding initial-boundary value problem for the Buckley-Leverett model shows that there are several different configurations of movement for oil, suspension, and mixture, depending on the initial and boundary data. We will prove that for all these configurations of initial and boundary data, the model describes unrealistic fluid motion. For example, the mixture begins to displace oil and suspension.

Keywords: Free boundary problems; Transport equations; Displacement of oil by suspension; Strong discontinuity conditions.

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1. Introduction

Free boundary problems for differential equations are some of the most difficult in the theory of partial differential equations. In these problems, along with solving differential equations, it is necessary to determine the domain in which this solution is sought. As a rule, this domain (boundary) is determined from an additional boundary condition at the free boundary. In the theory of free boundary problems, the Stefan problem, the Muskat problem, and the Hele-Shaw problem or the heat or Laplace equations are well known.[1-3] These problems are formulated quite simply, but so far the existence of a classical solution has been proven only locally in time (excluding some simple cases). As for systems of differential equations, here we should note the works of V. A. Solonnikov for free boundary problems to the Navier-Stokes system and A. Friedman.[4-6]

Note that the one-phase Stefan problem has a time monotone free boundary, which allowed authors to prove the existence of a classical solution for the one-phase Stefan problem arbitrary time interval.[1] However, in all other cases, including two-phase only the local existence of a solution can be proven. Separately, there is a large class of free boundary problems for the equations of gas dynamics and hydrodynamics of an ideal incompressible fluid. These problems are well studied and have a rich history.[7,8]

We will consider the Buckley-Leverett model, formulated in Refs. [9,10] and describing the displacement of oil by a suspension in the pore space of the absolutely rigid solid

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skeleton at the macroscopic level. Let us recall that such models as are called ***Phenomenological***, which means that all these models are nothing more than some set of axioms (postulates).[10-12]

Unlike ***Microscopic Mathematical Models***, with characteristic size is approximately ten of microns, in ***Macroscopic Mathematical Models*** the characteristic size is approximately meters or tens of meters. Because of this, these models do not distinguish between the microstructure of a continuous medium, since in such a model at each point the medium contains solid skeleton, fluids in the pores of this skeleton and ***Free (Unknown) Boundary***, separated these liquids.

All such models are built on the same principle. Fluid dynamics is usually controlled by the Darcy's system of filtration, or some modification of it. The equations describing the migration of oil and suspension simply postulated and, roughly speaking, are some modifications of continuity equations for the corresponding concentrations of oil and suspension. Finally, the difference of pressers in oil and in suspension is some known function of the oil concentration.

Exactly here there is a great variety of models, depending on the tastes and preferences of its authors. It is quite understandable, since the main mechanism of the physical process is focused on the unknown (free) boundary between different fluids and not spelled out in any way in the proposed macroscopic models.

The existence and uniqueness of a generalized solution to the system of Buckley-Leverett equations for smooth data of the problem was proved by S. Antontsev and V. Monakhov.[13] We will be interested in the structure of the weak solution to the Buckley-Leverett system of equations for a discontinuous initial oil concentration. In the terminology of L. V. Ovsyannikov, such a problem is called ***The problem of the decay of a strong discontinuity***.[14] The Buckley-Leverett model and its analogs are phenomenological mathematical models and serve as the basis model for existing hydrodynamic simulators, such as ***Eclipse***, ***Black Oil*** (Schlumberger), ***Tempest*** (Roxar), ***VIP*** (Landmark) and ***TimeZYX*** (Standard Oil and Trust).

Note that a hydrodynamic simulator is a certain ***Scale*** (set) of mathematical models of an oil reservoir of varying degrees of accuracy, supplemented with digital characteristics of physical properties, such as density and elastic properties of the solid skeleton, soil, density and viscosity of filtered liquids, as well as geometric characteristics of the reservoir in consideration, such as the structure of the solid skeleton, the geometry of the domain occupied by the reservoir, and visualization programs for numerical implementations.

Existing simulators, according to their purpose, must adequately reflect the simulated physical process.

Do existing simulators solve this problem?

Let's reformulate the question differently.

Since the basis of any hydrodynamic simulator is the corresponding scale of mathematical models (ideally!), the question can be formulated as follows:

Do the existing mathematical models underlying existing hydrodynamic simulators adequately reflect the physical process being modeled?

Only adequate mathematical models can optimize the oil production process, and only with adequate modeling can the main problem of a hydrodynamic simulator be solved - this is, of course, obtaining ***Maximum benefit from the exploitation of the field***.

A positive or negative answer depends on what exactly ***Adequacy of a mathematical model for a given physical process means***.

To do this, it is necessary to formulate ***Adequacy Criteria of the mathematical model***.

In the case of phenomenological models, the criterion of adequacy can only be experiment. Is experiment a criterion of adequacy?

The answer is No.

In fact, it makes no sense to talk about an experiment, since in any phenomenological model there are enough free parameters and even functions that are in no way related to the geometry of the reservoir (porosity and structure of the pore space) or to the physical characteristics of the displacement process (viscosity and density of filtered liquids and density and elastic properties of the solid skeleton). Therefore, by varying the indicated constants and functions, one can achieve agreement with any experiment!

Let us recall that, by definition, any phenomenological mathematical model is a set of postulates (axioms) expressed using differential equations, supplemented by appropriate boundary and initial conditions, as well as defining relations.

In this case, the characteristic dimensions in macroscopic models are meters or tens of meters. Because of this, such models do not distinguish between the microstructure of the continuous medium, nor the free boundary separating the liquids, nor the features of the interaction of liquids with the solid skeleton of the soil (adhesion or sliding conditions), since in such a model at each point of the continuous medium there are rock (solid skeleton) together with liquid in the pores of this skeleton, and a free boundary separating the various components of the medium.

All such models are built on the same principle. The dynamics of the fluid are usually controlled by the Darcy filtration equations or some modification of it, and the interaction of fluids is regulated by the laws of conservation of mass for each fluid. But all the fundamentally important changes occur precisely at the microscopic level, corresponding to the average size of pores or cracks in rocks, while any of the proposed macroscopic models operate on completely different (orders of magnitude larger) scales, which explains their diversity. The authors of such models simply do not have either an accurate method for describing physical processes at the microscopic level based on the fundamental laws of Newtonian continuum mechanics, or the ability to take into account the microstructure of rocks in macroscopic models. Therefore, they have to limit themselves to some speculative considerations (postulates) formulated by the authors themselves. Recent research efforts have aimed to overcome these limitations by employing more rigorous numerical and data-driven methods to simulate wave propagation in heterogeneous porous and composite media.[15-19]

In view of the above, a natural question arises: if there are several macroscopic models describing the same physical process, which of them most adequately reflects this process? Where is the criterion of truth here?

The answer to this question is quite complex and is beyond the scope of this article. Let's just say that in order to derive a macroscopic model adequate to the physical process under consideration, it is first necessary, following the principles formulated in the works of J. B. Keller and E. Sanchez-Palencia,[20,21] to describe this process based on the equations of Newton's classical mechanics at the microscopic level (average size of tens of microns) and only then, using mathematically strict homogenization, derive a macroscopic model that most accurately describes this physical process. The analysis is carried out within the framework of Sobolev spaces and classical results from functional analysis, which provide the necessary tools for establishing existence and uniqueness of weak solutions.[22-24]

2. Problem statement

We consider one spatial variable x and look for the solution to the Buckley-Leverett system in the domain $\Omega_T = \Omega \times (0,T)$, $\Omega = (0,1) \subset R = (-\propto, \propto)$, consisting of Darcy's system of filtration.

$$v_{ol} = -\frac{k}{\mu_{ol}} f_{ol}(c) \frac{\partial p_{ol}}{\partial x} \tag{1}$$

$$v_{sp} = -\frac{k}{\mu_{sp}} f_{sp}(c) \frac{\partial p_{sp}}{\partial_x} \tag{2}$$

and laws of conservation of mass in the domain Ω_T.

$$\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x} = 0 \tag{3}$$

$$\frac{\partial}{\partial t}(m(1-c) + \frac{\partial v_{sp}}{\partial x} = 0 \tag{4}$$

The system Eqs. (1)-(4) is completed with the state equations

$$kp_{ol} - kp_{sp} = p_{cap} = \alpha_{cap} c \tag{5}$$

$$f_{ol}(c) = \alpha_{ol} c \tag{6}$$

$$f_{sp}(c) = \alpha_{sp}(1-c) \tag{7}$$

and the following boundary and initial conditions.

$$v_{sp}(x,0) = v_{ol}(x,0) = 0, c(x,0) = c^0(x), c^0(x) = 0, \text{for } 0 < x < \tfrac{1}{2}; c^0(x) = 1, \text{for } \tfrac{1}{2} < x < 1 \tag{8}$$

In Eqs. (1)-(8) c is a concentration of oil in the pore liquid, $(1-c)$ is a concentration of suspension in the pore liquid, v_{ol} is the oil velocity, v_{sp} is the suspension velocity, p_{ol} is the oil pressure, v_{sp} is the suspension pressure, v_{cap} is *Capillary Pressure*, k is the dimensionless permeability, m is the porosity, μ_{ol} is the dimensionless oil viscosity and μ_{sp} is the dimensionless suspension viscosity.

The relation Eq. (5) means that capillary pressure p_{cap} depends only on the concentration c of the oil.

Positive constants μ_{ol}, μ_{sp}, α_d, α_{sp} and α_{cap} are supposed to be known.

First of all, using the obvious consequence of the sum of Eqs. (3) and (4).

$$\frac{\partial}{\partial t}(v_{ol} + v_{sp}) = 0, \text{or } v_{ol} + v_{sp} = \pm|V(t)| \tag{9}$$

We transform Eqs. (1)-(6) in the domain $\Omega_{mx}(t)$ to a form convenient for us:

$$\frac{f_{ol}}{\mu_{ol}} k \frac{\partial p_{ol}}{\partial x} + \frac{\partial p_{sp}}{\mu_{sp}} k \frac{\partial p_{sp}}{\partial x} = \mp|V| \tag{10}$$

$$\frac{f_{ol}}{\mu_{ol}}\left(k \frac{\partial p_{sp}}{\partial x} + \alpha_{cap} \frac{\partial c}{\partial x}\right) + \frac{f_{sp}}{\mu_{sp}} k \frac{\partial p_{sp}}{\partial x} = \mp|V| \tag{11}$$

$$k \frac{\partial p_{sp}}{\partial x} = \mp|V| \frac{\mu_{ol} f_{sp}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} - \frac{\alpha_{cap} \mu_{sp} f_{ol}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} \frac{\partial c}{\partial x} \tag{12}$$

$$k \frac{\partial p_{ol}}{\partial x} = \mp|V| \frac{\mu_{sp} f_{ol}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} + \frac{\alpha_{cap} \mu_{ol} f_{sp}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} \frac{\partial c}{\partial x} \tag{13}$$

$$v_{ol} = -\frac{f_{ol}}{\mu_{ol}} k \frac{\partial p_{ol}}{\partial x} = \pm|V| \frac{\mu_{sp} f_{ol}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} - \varphi(c) \frac{\partial c}{\partial x} \tag{14}$$

$$v_{sp} = -\frac{f_{sp}}{\mu_{sp}} k \frac{\partial p_{sp}}{\partial x} = \pm|V| \frac{\mu_{ol} f_{sp}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} + \varphi(c) \frac{\partial c}{\partial x} \tag{15}$$

$$\varphi(c) = \frac{\alpha_{cap} f_{ol} f_{sp}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} \tag{16}$$

$$m \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\mp|V| \frac{\mu_{sp} f_{ol}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}} + \varphi(c) \frac{\partial c}{\partial x}\right) \tag{17}$$

The sign in the first summand of the right-hand side in Eq. (17) does not play a role because of equality.

$$\frac{\partial}{\partial x}\left(\frac{\mu_{ol} f_{sp}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}}\right) = \frac{\partial}{\partial x}\left(1 - \frac{\mu_{sp} f_{ol}}{\mu_{sp} f_{ol} + \mu_{ol} f_{sp}}\right) \tag{18}$$

Taking into account equalities Eq. (6) we get

$$m \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(\mp|V(t)| \frac{\mu_{sp} \alpha_{ol} c}{\mu_{sp} f_{ol}(c) + \mu_{ol} f_{sp}(c)} + \varphi(c) \frac{\partial c}{\partial x}\right) \tag{19}$$

Let $S_0 = \{x \in \mathit{\Omega}: x = 0\}$, $S_1 = \{x \in \mathit{\Omega}: x = 1\}$, $\mathit{\Gamma}^0(t) = \{t \in (0, t_0): x = R^0(t)\}$ be a boundary between $\mathit{\Omega}_{sp}^0(t) = \{x \in \mathit{\Omega}: 0 < x < R^0(t)\}$ and $\mathit{\Omega}_{ol}^0(t) = \{x \in \mathit{\Omega}: 0 < R^0(t) < x < 1\}$, $\mathit{\Gamma}^-(t) = \{t \in (0, t_0): x = R^-(t)\}$ be the boundary between $\mathit{\Omega}_{sp}(t) = \{x \in \mathit{\Omega}: 0 < x < R^-(t)\}$, $\mathit{\Omega}_{mx}(t) = \{x \in \mathit{\Omega}: R^-(t) < x < R^+(t)\}$ and $\mathit{\Gamma}^+(t) = \{t \in (0, t_0): x = R^+(t)\}$ be the boundary between $\mathit{\Omega}_{mx}(t)$ and $\mathit{\Omega}_{ol}(t) = \{x \in \mathit{\Omega}: R^+(t) < x < 1\}$. We also put $\mathit{\Omega}_{ol,t_0}^0 = \cup_{t=0}^{t_0} \mathit{\Omega}_{ol}^0(t)$, $\mathit{\Omega}_{sp,t_0} = \cup_{t=0}^{t_0} \mathit{\Omega}_{sp}(t)$, $\mathit{\Omega}_{mx,t_0} = \cup_{t=0}^{t_0} \mathit{\Omega}_{mx}(t)$ and $\mathit{\Gamma}_{t_0}^{\pm} = \cup_{t=0}^{t_0} \mathit{\Gamma}^{\pm}(t)$.

In this publication we will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity of the initial concentration of oil, when at rest in one half $\Omega_{sp}(0) = \{x \in \Omega = (0,1): 0 < x < \frac{1}{2}\}$ of the domain Ω there is suspension, and in the other half of the domain $\Omega_{ol}(0) = \{x \in \Omega = (0,1): \frac{1}{2} < x < 1\}$ there is an oil, and these domains are separated by an impenetrable partition $\Gamma(0) = \{x \in \Omega: x = \frac{1}{2}\}$.

At the initial moment of time, the partition is removed and between suspension in the domain $\Omega_{sp}(0)$ and oil in the domain $\Omega_{ol}(0)$ immediately appears a zone of mixing of oil and suspension. We will analyze all possible configurations of solutions for all possible boundary and initial conditions.

For the case of no mixture and the existence of only one free boundary $x = R(t)$ there are two cases.

Cnf1: the suspension in the domain $\Omega^0_{sp,t_0} = \cup_{t=0}^{t_0} \Omega^0_{sp}(t)$, $\Omega^0_{sp}(t) = \{x \in \Omega: 0 < R(t) < x < 1\}$ displace the oil in the domain $\Omega^0_{ol,t_0} = \cup_{t=0}^{t_0} \Omega^0_{ol}(t)$, $\Omega^0_{ol}(t) = \{x \in \Omega = (0,1): R(t) < x < 1\}$ until to the moment t_0 when $\Omega^0_{sp}(t_0) = \Omega$.

Cnf2: the oil in the domain Ω^0_{ol,t_0} displace the suspension in the domain Ω^0_{sp,t_0} until to the moment t_0 when $\Omega^0_{ol}(t_0) = \Omega$.

We will prove that configurations *Cnf1* and *Cnf2* are impossible.

The other configurations are only possible for two free boundaries $R^{\pm}(t)$, when between the domain $\Omega_{sp}(0) = \{x \in \Omega = (0,1): 0 < x < \frac{1}{2}\}$, occupied by suspension and domain $\Omega_{ol}(0) = \{x \in \Omega = (0,1): \frac{1}{2} < x < 1\}$, occupied by the oil, immediately arises a zone $\Omega_{mx}(t)$ of a mixture of oil and suspension.

Here we have five different configurations.

For the sake of simplicity we put $R^-(t) = 0$ for $t > t^-$ if $R^-(t) > 0$ for $0 < t < t^-$ and $R^-(t^-) = 0$ if $R^-(t)$ is decreasing function and $R^+(t^+) = 1$ for $t > t^{\pm}$ if $R^+(t) < 1$ for $0 < t < t^+$ and $R^+(t^+) = 1$ if $R^+(t)$ is increasing function.

Cnf3: the mixture in the domain $\Omega_{mx,t_0} = \cup_{t=0}^{t_0} \Omega_{mx}(t)$, $\Omega_{mx}(t) = \{x \in \Omega = (0,1): R^-(t) < x < R^+(t)\}$ begins to displace the suspension in the domain $\Omega_{sp,t_0} = \cup_{t=0}^{t_0} \Omega_{sp}(t)$, $\Omega_{sp}(t) = \{x \in \Omega = (0,1): 0 < x < R^-(t)\}$ and oil in the domain $\Omega_{ol,t_0} = \cup_{t=0}^{t_0} \Omega_{ol}(t)$, $\Omega_{ol}(t) = \{x \in \Omega = (0,1): R^+(t) < x < 1\}$ until to the moment t_0 when $\Omega_{mx}(t_0) = \Omega$;

Cnf4: the mixture in the domain Ω_{mx,t_0} begins to displace the suspension in the domain Ω_{sp,t_0} and oil in the domainΩ_{ol,t_0} until to the moment t_* and then the oil begins to display the mixture up to moment t_0 up to the moment t_0 when $\Omega_{ol}(t_0) = \Omega$;

Cnf5: the mixture in the domain Ω_{mx,t_0} begins to displace the oil in the domain Ω_{ol,t_0} and suspension in the domain Ω_{sp,t_0} until to the moment t_* and then the suspension begins to display the mixture and oil up to moment t_0 when $\Omega_{sp}(t_0) = \Omega$;

Cnf6: the oil in the domain Ω_{ol,t_0} begins to displace the mixture in the domain Ω_{mx,t_0} and suspension in the domain Ω_{sp,t_0} up to the moment t_0 when $\Omega_{ol}(t_0) = \Omega$;

Cnf7: the suspension in the domain Ω_{sp,t_0} begins to displace the mixture in the domain Ω_{mx,t_0} and oil in the domain Ω_{ol,t_0} up to the moment t_0 when $\Omega_{sp}(t_0) = \Omega$.

All these situations are realized for boundary conditions, listed below.

First two cases with one free boundary in principle be possible if $U_{sp}(0,t) = |V(t)|$, $U_{ol}(1,t) = |V(t)|$, $0 < t < t_0$ for *Cnf1* and $U_{sp}(0,t) = -|V(t)|$, $U_{ol} = -|V(t)|$, $0 < t < t_0$ *for Cnf2*.

For the case of two free boundaries there are five different combinations of boundary conditions:

Cnf3: $U_{sp}(0,t) = -|V(t)|, U_{ol}(1,t) = |V(t)|$, $V_{sp} = -|V(t)|, V_{ol} = |V(t)|$;

Cnf4: $U_{sp}(0,t) = -|V(t)|, U_{ol}(1,t) = |V(t)|$ up to time moment t_* and then $U_{sp}(0,t) = |V(t)|, U_{ol} = |V(t)|$;

Cnf5: $U_{sp}(0,t) = -|V(t)|, U_{ol}(1,t) = |V(t)|$ up to time moment t_* and then $U_{sp}(0,t) = -|V(t)|, U_{ol} = -|V(t)|$ up to time moment t_* and then $U_{sp}(0,t) = |V(t)|, U_{ol} = |V(t)|$;

Cnf6: $U_{sp} = -|V(t)|, U_{ol} = -|V(t)|$;

Cnf7: $U_{sp}(0,t) = |V(t)|, U_{ol}(1,t) = |V(t)|$.

The functions $R(t)$ and $R^{\pm}(t)$ satisfy the following initial condition

$$R(0) = R^{\pm}(0) = \frac{1}{2} \tag{20}$$

and are assumed to be unknown *Free boundaries*.

Finally, we note that the motion of liquids with real surface tension in pipette droplets is different from that described by configurations *Cnf3*-*Cnf7*: in the Buckley-Leverett model, the boundary $x = R^-(t)$ changes its position in time, while in the real motion of suspension droplets this boundary is a fixed line $x = \frac{1}{2}$ for $0 < t < t_0$ (see Figs. 1-5).

In the present manuscript we analyze all possible structures of displacement of oil by suspension which always realized by appearance of the mixture of oil and suspension with moving boundaries, separated suspension, mixture and oil and prove that for the boundary R-(t) between suspension and mixture for configuration *Cnf7*, that describe the displacements of oil by suspension, is moving boundary. This fact contradicts a real situation, such as the displacement of one liquid by another in a pipette in the presence of real surface tension, given the real curvature of the droplet (Figs. 1-5).

Fig. 1 shows system of real pipettes for different time moments (Case I). The initial stage of the oil displacement process after removing the impermeable partition. The suspension begins to move into the region previously occupied by oil, initiating the formation of a fluid interface. Fig. 2 shows system of real pipettes for different time moments (Case II).

Intermediate stage of the process showing the progressive movement of the suspension front and the initial development of the mixing zone between oil and suspension. Fig. 3 shows system of real pipettes for different time moments (Case III). Further evolution of the interface illustrating the coexistence of three regions—pure oil, pure suspension, and the mixture zone separating them. Fig. 4 shows system of real pipettes for different time moments (Case IV). Advanced stage of the displacement process where the curvature of the interface becomes stable due to surface tension effects, and the motion of the boundary slows down. Fig. 5 shows temporal evolution of fluid interfaces in a real pipette system (Final stage). The final configuration demonstrates a stationary boundary separating oil, suspension, and mixture, highlighting the discrepancy between the physical process and the Buckley–Leverett model predictions. Throughout the text we use the notations for functional spaces and norms in these spaces adopted.[23,24]

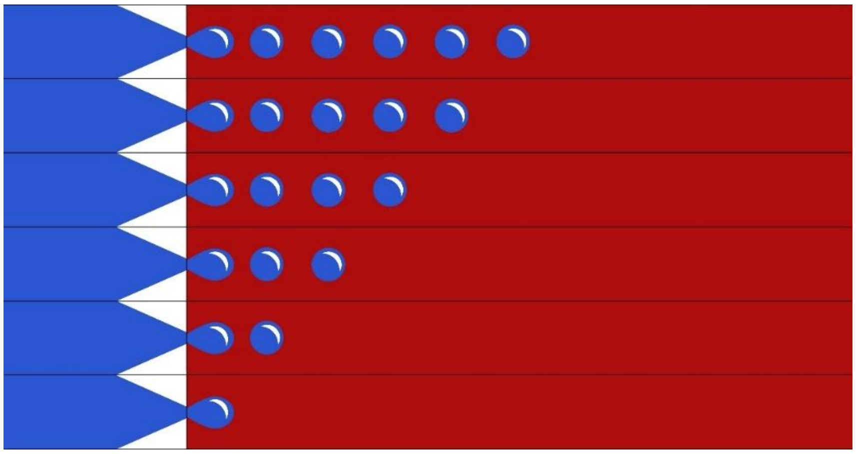

Fig. 1: System of real pipettes for different time moments. Case I.

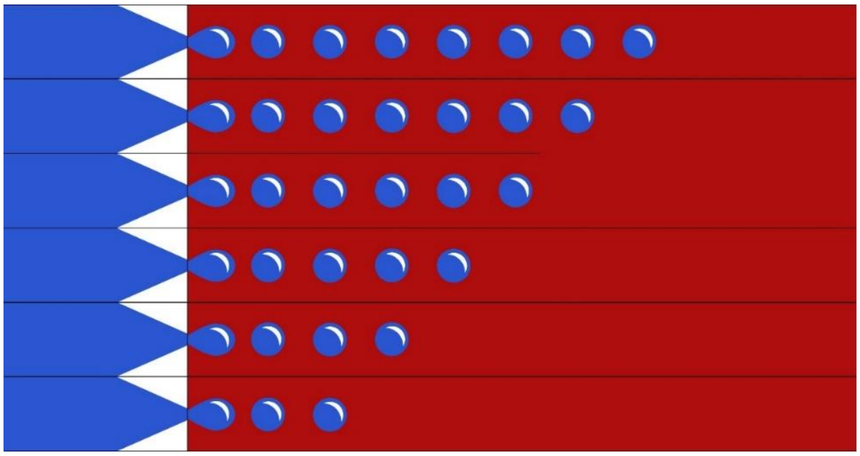

Fig. 2: System of real pipettes for different time moments. Case II.

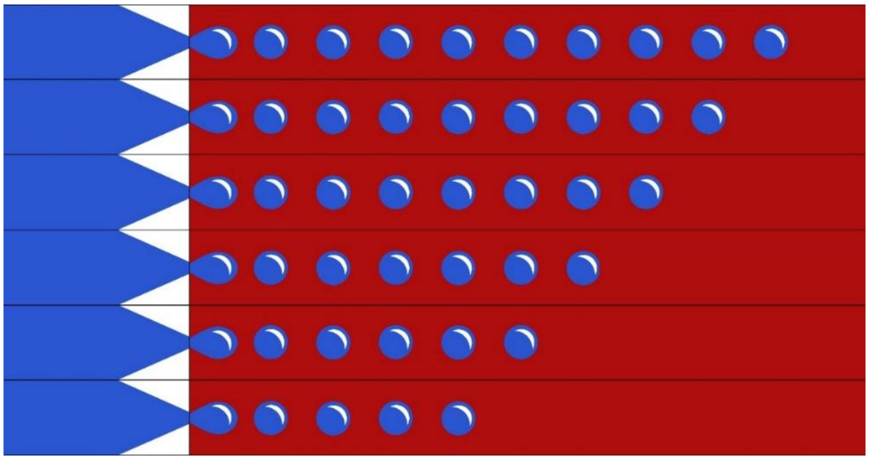

Fig. 3: System of real pipettes for different time moments. Case III.

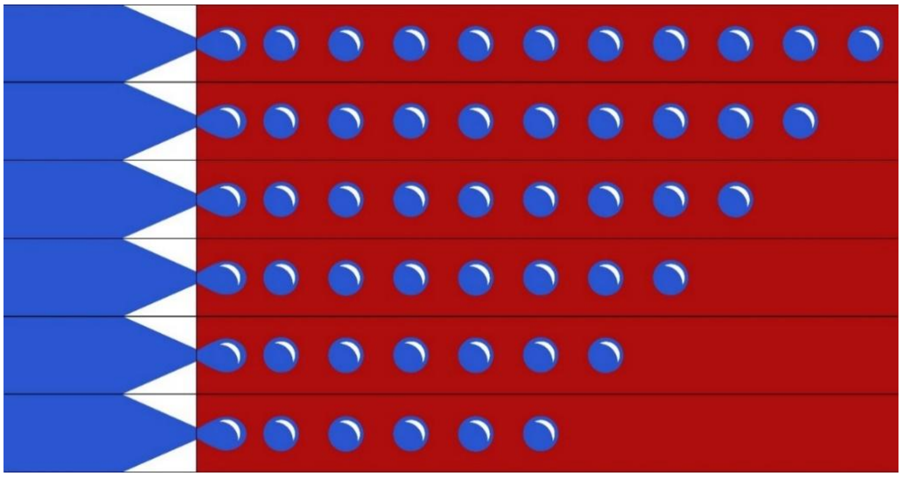

Fig. 4: System of real pipettes for different time moments. Case IV.

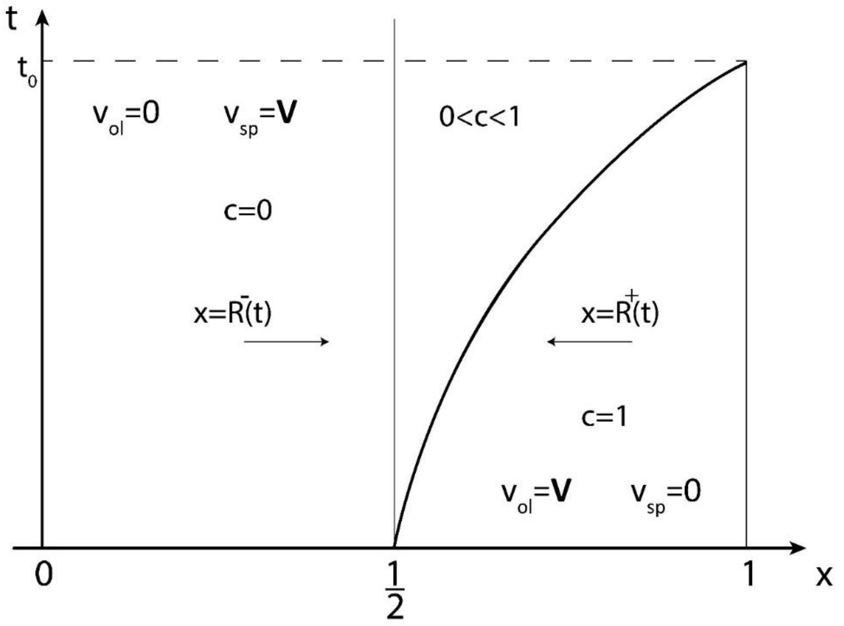

Fig. 5: Temporal evolution of fluid interfaces in a real pipette system (Final stage).

3. Experimental

Note that in the actual physical process of oil displacement by suspension, the suspension velocity v_{sp} and oil velocity v_{ol} are positive functions. Therefore, we will further assume that for the corresponding configurations, $V(t)$ is smooth function and

$$V \in \mathbb{H}^{\frac{2+\alpha}{2}}[0, t_0],\ \frac{m}{2} < \mu_{sp}^{-1}\alpha_{cap}\alpha_{sp} = V_0 \le V(t) \le V_1 \qquad (21)$$

Let

$$I^1 = \int_0^{t_0}\int_0^{R(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt \quad (22)$$

$$I^2 = \int_0^{t_0}\int_0^{R(t)}\left(\left(m(1-c)\frac{\partial \eta}{\partial t}\right) + v_{sp}\frac{\partial \eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\left(m(1-c)\frac{\partial \eta}{\partial t}\right) + v_{ol}\frac{\partial \eta}{\partial x}\right)dxdt \quad (23)$$

Definition 3.1

We say that the triple of functions $\{c, v_{ol}, v_{v_{sp}}\}$ is a weak solution to the problem Eqs. (1)-(8) for configuration Cnf1 if the functions $c, v_{ol}, v_{v_{sp}} \in \mathbb{W}_2^{1,0}(\Omega_{mx,t_0})$ satisfy the condition of the configuration Cnf1 together with the integral identities.

$$\int_0^{t_0}\int_0^{R(t)}\left(m(1-c)\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(mc\frac{\partial \eta}{\partial t} + v_{ol}\frac{\partial \eta}{\partial x}\right)dxdt = 0 \qquad (24)$$

$$\int_0^{t_0}\int_0^{R(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt = 0 \qquad (25)$$

which hold true for any smooth functions η, vanishing at the boundary $\partial\Omega$ and at time moments $t = 0$ and $t = t_0$.

Theorem 3.1

1) The problem Eqs. (1)-(8) for configurations Cnf1 has an

unique weak solution $c = 0, v_{ol} = 0, v_{sp} = V$ in the domain Ω^0_{sp}, $c = 1, v_{ol} = V, v_{sp} = 0$ in the domain Ω^0_{sp} and $R(t) = \frac{1}{2} + \int_0^t V\tau d\tau$ for $0 < t < T$.

2) The time moment T is defined as a moment, when $\Omega_{sp}(T) = \Omega$ as $R(t) = \frac{1}{2} + \int_0^T V\tau d\tau$.

Definition 3.2

We say that the triple of functions $\{c, v_{ol}, v_{sp}\}$ is a weak solution to the problem Eqs. (1)-(8) for configuration Cnf2 if the functions $c, v_{ol}, v_{sp} \in \mathbb{W}^{1,0}_2(\Omega_{mx,t_0})$ satisfy the condition of the configuration Cnf2 together with the integral identities Eqs. (24) and (25).

Theorem 3.2

1) The problem Eqs. (1)-(8) for configurations Cnf2 has an unique weak solution $c = 0, v_{ol} = 0, v_{sp} = -V$ in the domain Ω^0_{sp}, $c = 1, v_{ol} = -V, v_{sp} = 0$ in the domain Ω^0_{sp} and $R(t) = \frac{1}{2} - \int_0^t V\tau d\tau$ for $0 < t < T$.

2) The time moment T is defined as a moment, when $\Omega_{sp}(T) = \Omega$ as $R(t) = \frac{1}{2} - \int_0^T V\tau d\tau$.

To formulate the definition of the weak solution to the problem Eqs. (1)-(8) for the configuration Cnf7 we consider the chain of integral identities, equivalent to the system of differential equations Eqs. (1)-(3) under assumptions $c = 0, v_{sp} = V, v_{ol} = 0$ if in Ω_{sp,t_0} and $c = 1, v_{sp} = 0, v_{ol} = V$ in Ω_{ol,t_0}.

Let

$$I^1_{sp} = I^1_{sp} = \int_0^{t_0}\int_0^{R^-(t)}\left(\frac{\partial}{\partial t}\big(m\eta(1-c)\big) + \frac{\partial}{\partial x}\big(\eta v_{sp}\big)\right)dxdt + \int_0^{t_0}\int_0^{R^-(t)}\left(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}\big(\eta v_{sp}\big)\right)dxdt \tag{26}$$

$$I^2_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\left(m(1-c)\frac{\partial\eta}{\partial t}\big(m\eta(1-c)\big) + \frac{\partial\eta}{\partial x}\big(\eta v_{sp}\big)\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}\big(\eta v_{sp}\big)\right)dxdt\right. \tag{27}$$

Then

$$\int_0^{t_0}\int_0^{R^-(t)}\eta\left(\frac{\partial}{\partial t}\big(m(1-c)\big) + \frac{\partial v_{sp}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\eta\left(\frac{\partial}{\partial t}\big(m(1-c)\big) + \frac{\partial v_{sp}}{\partial x} dxdt = I^1_{sp} - I^2_{sp} = 0\right. \tag{28}$$

Definition 3.3

We say that the set of functions $\{c, v_{sp}, v_{mx}, v_{ol}, p_{sp}, p_{ol}\}$ is a weak solution to the problem Eqs. (1)-(8) for configuration *Cnf7*, if functions v_{sp} and v_{ol} belong to the space $\mathbb{W}^{1,0}_2(\Omega_{mx,t_0})$, $c \in \mathbb{L}_\infty\big(0, T; \mathbb{BV}(\Omega)\big)$, $\frac{\partial c}{\partial t} \in \mathbb{L}_\infty\left(0, T; \mathbb{BV}^{-1}(\Omega_{mx,t_0})\right)$ and hold true integral identities,

$$\int_0^{t_0}\int_0^{R^-(t)}\left(m\frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}(m(1-c)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x})dxdt = 0 \tag{29}$$

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^+(t)}^{1}\left((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt = 0 \tag{30}$$

which hold true for any smooth functions η, vanishing at the boundary $\partial\Omega$ and at time moments $t = 0$ and $t = t_0$.

Remark 3.1

For definition of the space $\mathbb{L}_\infty\big(0, T; \mathbb{BV}^{-1}(\Omega)\big)$ (chapter 1, section 1.2).[25]

Theorem 3.3

Let

$$V(t) > \alpha_{cap}\alpha_{sp}\mu_{sp}^{-1} = V_0 = \text{const} > \frac{m}{2} \quad \text{for all } 0 \le t \le t_0 \tag{31}$$

Then the problem Eqs. (1)-(8) for the configuration *Cnf7* has an unique weak solution up to time moment T, where $\Omega_{sp}(T) = \Omega$ (the suspension completely displaced the oil).

Theorem 3.4

Under conditions of Theorem 2.2 the solution to the problem Eqs. (1)-(8) for the configuration Cnf7 Eqs. (1)-(8) is infinitely smooth in the domain $\Omega_{[t_0,T]}$ for $t_* > 0$.

4. Results and discussion

Proof of Theorems 2.1 and 2.2. Let functions c, v^0_{sp}, v^0_{ol} be solution to the problem Eqs. (1)-(8) for the configuration *Cnf1* in the domains $\Omega^0_{sp,t_0} = \bigcup_{t=0}^{t=t_0}\Omega^0_{sp}(t)$ and $\Omega^0_{ol,t_0} = \bigcup_{t=0}^{t=t_0}\Omega^0_{ol}(t)$ be solution to the problem and η be some arbitrary function, vanishing at the boundary $\partial\Omega$ and $t = 0$.

Then hold true the following chain of integral identities

$$\int_0^{t_0}\int_0^{R(t)}\eta\left(\frac{\partial}{\partial t}\big(m(1-c)\big) + \frac{\partial v_{sp}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\eta\left(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\right)dxdt + \int_0^{t_0}\int_0^{R(t)}\left(m(1-c)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(mc\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt = I^1 - I^2 = 0 \tag{32}$$

where

$$I^1 = \int_0^{t_0}\int_0^{R(t)}\left(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}\big(\eta v_{sp}\big)\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt = 0 \tag{33}$$

$$I^2 = \int_0^{t_0}\int_0^{R(t)}\left(\big(m(1-c)\big)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\big(m(1-c)\big)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt = 0 \tag{34}$$

The equality $I^2 = 0$, which is a definition of the weak solution to the configuration *Cnf2*, implies equality $I^1 = 0$.

Next we will use identity $I^1 = 0$ to find the proper free boundary $x = R(t)$ and, if it possible, to calculate boundary condition at the free boundary. Usually for that is necessary two conditions: one for the definition the free boundary and the another for corresponding boundary condition.

We have the following chain of equalities:

$$I^1 = \int_0^{t_0}\int_0^{R(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt + \int_0^{t_0}\int_{R(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt = \int_0^{t_0}\left(\frac{d}{dt}\int_0^{R(t)}\eta m dx - \frac{dR}{dt}(t_0)\eta(R(t_0,t_0))m + \frac{d}{dt}\int_{R(t)}^{1}\eta m dx + \frac{dR}{dt}(t_0)\eta(R(t_0,t_0)mdx\right)dt = 0 \tag{35}$$

Thus, the identity $I^1 = 0$ does not contain any additional information, and the only way to solve the problem is postulate

$$\frac{dR}{dt} = V(t), R(t) = \frac{1}{2} + \int_0^t V\tau d\tau \tag{36}$$

The configuration Cnf2 with $c = 0, v_{sp}^0 = -V(t)$ in the domain Ω_{sp,t_0}^0 and $c = 1, v_{ol}^0 = -V(t)$ in the domain Ω_{ol,t_0}^0 is treated in a similar way and expressed by equalities

$$\frac{dR}{dt} = -V, R(t) = \frac{1}{2} - \int_0^t V\tau d\tau \tag{37}$$

Proof of Theorem 2.3. For simplicity, we restrict ourselves to the configuration *Cnf7* with two free boundaries $x = R^{\pm}(t)$, $0 \le R^-(t) \le R^+(t) \le 1$. All other configurations are treated similarly.

4.1 Derivation of boundary conditions on the free boundary $\boldsymbol{\Gamma^-(t)}$

Let η be some arbitrary smooth function vanishing at S_0, $\Gamma^+(t)$, $t = 0$ and $t = t_0$, $V(t)$ be smooth positive function with respect to time, c be smooth function in the domain Ω_{mx,t_0}, $0 < c^-(t) \le c \le c^+(t) < 1$, $v_{sp}(t) = V$, $v_{ol}(t) = 0$, $c = 0$ in the domain Ω_{sp,t_0}, $v_{sp} = 0$, $v_{ol} = V(t)$, $c = 1$ in the domain Ω_{ol,t_0} and v_{ol} and v_{sp} be smooth functions defined in the domain Ω_{mx,t_0} by relations Eqs. (14) and (15):

$$v_{ol} = V\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} - \varphi(c)\frac{\partial c}{\partial x} = f_{ol}\frac{V\mu_{sp}-\alpha_{cap}\alpha_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + f_{ol}\frac{\alpha_{cap}\alpha_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}c\frac{\partial c}{\partial x} \ge 0, 0 \le t \le t_0 \tag{38}$$

$$v_{sp} = V\frac{\mu_{ol}f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + \varphi(c)\frac{\partial c}{\partial x} = f_{sp}\frac{V\mu_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + f_{sp}\frac{\alpha_{cap}\alpha_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}c\frac{\partial c}{\partial x} \ge 0, 0 \le t \le t_0 \tag{39}$$

Let also

$$I_{sp}^1 = \int_0^{t_0}\int_0^{R^-(t)}\left(m\frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(m\frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x}\right)dxdt \tag{40}$$

$$I_{sp}^2 = \int_0^{t_0}\int_0^{R^-(t)}\left(\frac{\partial}{\partial t}(m\eta) + \frac{\partial}{\partial x}(\eta V)\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt \tag{41}$$

Then we have

$$\int_0^{t_0}\int_0^{R^-(t)}\eta\left(\frac{\partial}{\partial t}m(1-c) + \frac{\partial v_{sp}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\eta\left(\frac{\partial}{\partial t}m(1-c) + \frac{\partial v_{sp}}{\partial x}\right)dxdt + \int_0^{t_0}\int_0^{R^-(t)}\left(m\frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(m(1-c)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x}\right)dxdt - \int_0^{t_0}\int_0^{R^-(t)}\left(\frac{\partial}{\partial t}(m\eta) + \frac{\partial}{\partial x}(\eta V)\right)dxdt - \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt = I_{sp}^1 - I_{sp}^2 = 0 \tag{42}$$

By construction $I_{sp}^1 = 0$ due to an arbitrary choice of the function η. Therefore the equality $I_{sp}^2 = 0$ implies $I_{sp}^1 = 0$, or $I_{sp}^1 = 0,\ I_{sp}^2 = 0$.

The first identity $I_{sp}^1 = 0$ in Eq. (42) with some additional conditions can serve as a definition of the weak solution of the problem Eqs. (1)-(8), and the second identity $I_{sp}^2 = 0$ in Eq. (42) gives us the equation for finding the unknown function $R^-(t)$.

Let $v_{sp}^- = v_{sp}(R^-(t)+0,t)$, $c^- = c(R^-(t)+0,t)$ and η vanishing at $\Gamma^+(t)$, $x = 0$ $t = 0$ and $t = t_0$.

Then equality $I_{sp}^2 = 0$ results

$$I_{sp}^2 = \int_0^{t_0}\int_0^{R^-(t)}\left(\frac{\partial}{\partial t}(m\eta) + V\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(m\eta(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt = \int_0^{t_0}\left(\eta\left(m + V + \frac{d}{dt}\int_{R^-(t)}^{R^+(t)} m\,\eta(1-c)dx -\right)dt + \eta^-(m(1-c^-)\frac{dR^-}{dt} - v_{sp}^-\right)dt = \int_0^{t_0}\eta^-\left(m + V + (1-c^-)m\frac{dR^-}{dt} - v_{sp}^-)\right)dt = \int_0^{t_0}\eta^-\left(m + (1-c^-)m\frac{dR^-}{dt} - v_{ol}^-)\right)dt = 0 \tag{43}$$

Because of the arbitrary choice of function η last equality implies

$$(1-c^-(t))m\frac{dR^-}{dt}(t) = v_{ol}(R^-(t+0),t) - m \tag{44}$$

To find the function $c^-(t)$ we put

$$I_{sp}^3 = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(mc\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt \tag{45}$$

$$I_{sp}^4 = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt \tag{46}$$

and consider the integral identity

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\eta\left(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x})dxdt - \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta m v_{ol})\right)dxdt = I_{sp}^3 - I_{sp}^4 = 0 \tag{47}$$

As before (see Eq. (42)) we conclude that $I_{sp}^3 = 0, I_{sp}^4 = 0$, where the first identity is the basis for the definition of the weak solution of the problem Eqs. (1)-(8), and the second identity gives us the equation for finding the unknown function $c^-(t)$.

In fact, let η be the same as before and $c = 0$, $v_{ol} = 0$, $v_{sp} = V(t)$ in Ω_{sp,t_0}. Then the second equality in Eq. (47)

gives us

$$I^4_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt = \int_0^{t_0}\left(\frac{d}{dt}\int_{R^-(t)}^{R^+(t)}\eta\, mcdx + \eta(R^-(t),t)(m\frac{dR^-}{dt}(t)c^-(t) - v_{ol}(R^-(t)+0,t)\right)dt = \int_0^{t_0}\eta\,(R^-(t),t)(m\frac{dR^-}{dt}(t)c^-(t) - v_{ol}(R^-(t)+0,t))dt \tag{48}$$

Thus, due to the arbitrary choice of functions η

$$c^-(t)m\frac{dR^-}{dt}(t) = v_{ol}(R^-(t)+0,t) \tag{49}$$

The sum of Eqs. (44) and (49) gives us

$$m\frac{dR^-}{dt}(t) = 2v_{ol}(R^-(t)+0,t) - m > 2V_0 - m > 0 \tag{50}$$

due to Eq. (21).

Substitution Eq. (50) into Eq. (49) results

$$c^-(t) = v_{ol}(R^-(t)+0,t)(2v_{ol}(R^-(t)+0,t) - m)^{-1} > 0 \tag{51}$$

4.2 Derivation of boundary conditions for strong discontinuities at free boundary $\boldsymbol{\Gamma^+(t)}$

Let η be some arbitrary smooth function vanishing at $\Gamma^-(t)$, S_1, $t = 0$ and $t = t_0$, $V(t)$ be smooth function with respect to time, defined by relation Eq. (49) c be smooth function in the domain Ω_{mx,t_0}, $v_{sp}(t) = 0$, $v_{ol}(t) = V(t)$, $c = 1$ in the domain Ω_{ol,t_0}, v_{ol} and v_{sp} be smooth functions defined in the domain Ω_{mx,t_0} by relations Eqs. (38) and (29)

$$I^1_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^+(t)}^{1}\left((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt \tag{52}$$

$$I^2_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial t}(\eta v_{ol})\right)dxdt + \int_0^{t_0}\int_{R^+(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial t}(\eta v_{ol})\right)dxdt \tag{53}$$

$$c^+(t) = c(R^+(t), t-0) \tag{54}$$

$$v_{ol}(R^+(t), t-0) = v^+_{ol}(t) \tag{55}$$

Then, we have

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\eta\left(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^+(t)}^{1}\eta\left(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt - \int_0^{t_0}\int_{R^+(t)}^{1}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt = I^1_{ol} - I^2_{ol} = 0 \tag{56}$$

As in the previous section, we obtain

$$I^1_{ol} = 0,\ I^2_{ol} = 0 \tag{57}$$

The last identity in Eq. (54) results

$$I^2_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt + I^2_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\right)dxdt + \int_0^{t_0}\left(\frac{d}{dt}\int_{R^-(t)}^{R^+(t)}(\eta mc)\,dx - \eta^+c^+m\frac{dR^+}{dt} + \eta^+v^+_{ol}\right)dt + \int_0^{t_0}\eta^+\,(-m-V)dt = \int_0^{t_0}\eta^+\left(-c^+m\frac{dR^+}{dt} + v^+_{ol} - m - V\right)dt \tag{58}$$

or

$$c^+(t)m\frac{dR^+}{dt}(t) = V(t) + m - v_{ol}(R^+(t)-0,t) = v_{sp}(R^+(t)-0,t) + m \tag{59}$$

To find $c^+(t)$ we put

$$I^3_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(m(1-c)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x}\right)dxdt \tag{60}$$

$$I^4_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt \tag{61}$$

and consider the integral identity

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\eta\left(\frac{\partial}{\partial t}m(1-c) + \frac{\partial v_{sp}}{\partial x}\right)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}(m(1-c)\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(\eta v_{sp})dxdt - \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt = I^3_{ol} - I^4_{ol} \tag{62}$$

As before we obtain

$$I^3_{ol} = I^4_{ol} = 0 \tag{63}$$

Thus

$$I^4_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)}\left(\frac{\partial}{\partial t}(\eta m(1-c)) + \frac{\partial}{\partial x}(\eta v_{sp})\right)dxdt = \int_0^{t_0}\left(\frac{d}{dt}\int_{R^-(t)}^{R^+(t)}\eta\, m(1-c)dx\right)dt + \int_0^{t_0}\eta^+\left(v^+_{sp} - (1-c^+)m\frac{dR^+}{dt}\right)dt \tag{64}$$

The last identity results:

$$(1-c^+)m\frac{dR^+}{dt}(t) = v_{sp}(R^+(t)-0,t) \tag{65}$$

The sum of Eqs. (56) and (61) give us

$$m\frac{dR^+}{dt}(t) = m + 2v_{sp}(R^+(t)-0,t) \tag{66}$$

After substitution Eqs. (62) into (56) we obtain

$$0 < c^+(t) = \frac{v_{sp}(R^+(t)-0,t)+m}{v_{sp}(R^+(t)-0,t)+2m} < 1 \tag{67}$$

Remark 4.1

We note that all transformations of differential equations are correct by virtue of the maximum principle and local estimates of solutions of parabolic equations (see, Chapter IV, §10).[25]

Lemma 4.1

Let c be solution to the initial boundary value problem consisting of Eq. (19) in the domain Ω_{mx,t_0} and boundary conditions

$$c(R^+(t),t) = c^+(t) \tag{68}$$

Then

$$c \in \mathbb{H}^{2+\alpha,\frac{2+\alpha}{2}}\left(\overline{\Omega}_{mx,[t_0,t^0]}\right), R^{\pm}(t) \in \mathbb{H}^{\frac{2+\alpha}{2}}[t_0,t^0]\ for\ 0 < t_0 < t^0 \leq T \tag{69}$$

Proof. By construction the concentration c satisfies integral identities Eqs. (29) and (30). That is c is a weak solution to the problem Eqs. (1)-(8).

5. Conclusion

The manuscript submitted for publication is devoted to the structure of all possible solutions of the Bakley-Leverett mathematical model describing oil displacement by a suspension in rock formations. The importance of this model is determined by the fact that it is the basis of almost all known hydrodynamic simulators of oil reservoirs. Therefore, the focus of the manuscript is on the corresponding initial-boundary value problems in the configurations *Cnf1* and *Cnf7*, which describe oil displacement by suspension. All other configurations are considered in a similar manner.

It should be noted that phenomenological models of the physical process of oil displacement by suspension differ from models of the same process derived from Newton's classical laws of a continuous medium by homogenization the corresponding mathematical models at the microscopic level and which do not raise doubts as to the correctness of the homogenized equations and boundary conditions, as well as their uniqueness.

We mentioned in the introduction that phenomenological models postulate both the differential equations themselves and the boundary conditions, and that the same physical process is often described by different authors using completely different models. Why is this possible? Only because such models do not take into account the characteristics of a continuous medium at the microscopic level. In particular, the physical and geometric properties of the solid skeleton of the soil and the physical properties of liquids in the pore space of the solid skeleton are not specified anywhere, as is done in accurate microscopic models.[20, 21]

In configuration *Cnf1*, there are only two components (suspension and oil), separated by a free (unknown) boundary $x = R(t)$. For these configurations, the only possible system of integral identities Eqs. (24) and (25) is derived, equivalent to the corresponding initial-boundary value problems of the mathematical model in the configuration Cnf1. A precise analysis of these identities leads to the only possible solution, given by the explicit formulas presented in the conditions of Theorem 2.1.

Similarly, an initial-boundary value problem describing the oil displacement process in configuration Cnf7 is derived. By definition of the configuration Cnf7, a suspension is fed into the domain $\Omega_{sp}(t)$ through injection wells at the boundary $x = 0$, and oil in the domain $\Omega_{ol}(t)$ is taken up by production wells at the boundary $x = 1$.

At the initial moment of time, a suspension begins to be fed through the injection wells at the boundaries S^0 into the domain $\Omega_{sp}(t)$ occupied by the suspension, the velocity of which is equal to $V(t)$. Since the oil concentration in the domain $\Omega_{sp}(t)$ is zero, it will be zero there at all times, and the suspension velocity will be equal to $V(t)$. Similarly, the oil concentration in the domain $\Omega_{ol}(t)$ will be equal to 1, and its velocity will be equal to $V(t)$ at all times. Finally, the domain $\Omega_{mx}(t)$, occupied by a mixture of oil and suspension, will always separate the regions $\Omega_{sp}(t)$ and $\Omega_{ol}(t)$. That is, at all times $0 < \mathrm{R}^-(\mathrm{t}) < \mathrm{R}^+(\mathrm{t}) < 1$.

The next step will be to determine the limiting value of the concentration $\mathrm{c}^-(\mathrm{t})$ at the boundary $\mathrm{x} = \mathrm{R}^-(\mathrm{t})$ as the point x from the domain $\Omega_{mx}(t)$ approaches the boundary $\mathrm{x} = \mathrm{R}^-(\mathrm{t})$, and the boundary $\mathrm{x} = \mathrm{R}^-(\mathrm{t})$ itself. To do this, we will use the integral identities Eqs. (29) and (30). Analysis of these identities gives us Eqs. (50) and (51), which uniquely determine the values of $\frac{\mathrm{dR}^-}{\mathrm{dt}}(\mathrm{t})$ and $\mathrm{c}^-(\mathrm{t}) = \mathrm{c}(\mathrm{R}^-(\mathrm{t}) + 0, \mathrm{t})$. In a manner completely analogous to the previous case, the values $\frac{\mathrm{dR}^+}{\mathrm{dt}}(\mathrm{t})$ and $\mathrm{c}^+(\mathrm{t})$ are determined by Eqs. (62) and (63).

In conclusion, we note that for the given values of $v_{sp}^-(t)$ and $v_{ol}^+(t)$, each subsequent step starting from the domain Ω_{sp,t_0}, uniquely determines the next values, since there is only one integral identity for its determination. Given a set of differential equations and equations of state, as well as boundary and initial conditions Eqs. (1)-(8), we are unable to determine the functions $v_{sp}^-(t)$ and $v_{ol}^+(t)$. That is, we can not prove the uniqueness of the problem Eqs. (1)-(8) for the configuration Cnf7. There are some principles to choose the most appropriate functions $v_{sp}^-(t)$ and $v_{ol}^+(t)$, which give minima to the Total entropy of the system Eqs. (1)-(8).[25] Unfortunately, under the conditions of the mathematical model Eqs. (1)-(8), we cannot define such a concept as entropy and, thus, determine the unique solution to the mathematical model Eqs. (1)-(8).

In real conditions of oil displacement from the pipette system, a situation is possible where the considered region Ω is a unit cube consisting of parallelepipeds of unit length and side widths of 2ε on the end sides, containing cylinders (pipettes) of unit length and radius ε. One quarter of the cube is occupied by suspension, the second quarter by oil, and the rest by air. At the initial moment of time, a suspension with a velocity of $V(t)$ is supplied from one end of the cube. Then, due to the capillary pressure given by Laplace's formula,[26] oil droplets appear in the air (see Figs. 1-4), forming an area occupied by a mixture of oil and suspension. The formal homogenization of this process (macroscopic description) is shown in Fig. 5 and demonstrates that the Buckley-Leverett mathematical model inadequately describes the actual physical process of oil displacement by suspension.

Conflict of Interest

There is no conflict of interest.

Supporting Information

Not applicable.

CRedit Statement

Anvarbek Meirmanov: Conceptualization, Methodology, Project Administration, Writing – Original Draft. **Marat Nurtas:** Supervision, Data Curation, Validation, Writing – Review & Editing. **Oleg Galtsov:** Software, Formal Analysis. **Aizhan Ydyrys:** Validation, Visualization. **Merey Kenzhebayeva**: Funding Acquisition, Resource.

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