



One-Dimensional Non-Linear Non-Stationary System of Boltzmann Moment Equations in the Fifth Approximation with a Macroscopic Boundary Condition

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Abstract

The present study is devoted to the one-dimensional system of Boltzmann moment equations in the fifth approximation and macroscopic boundary conditions. The microscopic Maxwell boundary condition for the one-dimensional Boltzmann equation was approximated and boundary conditions for the one-dimensional non-linear non-stationary Boltzmann moment system in the fifth approximation were derived. The initial boundary value problems for the fifth approximation of the Boltzmann moment system with macroscopic boundary conditions were formulated. The system of Boltzmann moment equations represents a non-linear symmetric system of hyperbolic equations. The first part substantiates the existence and uniqueness of the solution to the initial boundary value problem for the one-dimensional system of Boltzmann moment equations in the fifth approximation with macroscopic boundary conditions in the space of functions that are continuous in time and square-summable over the spatial variable. In the second part, using a numerical method, an approximate solution to the initial boundary value problem for the system of Boltzmann moment equations is determined. The differential problem is approximated by a finite-difference scheme, and an algorithm and program for numerical implementation on a computer are compiled. At the end of the article, tables of values and graphs of the moments of the particle distribution function corresponding to the fifth approximation of the moment equations are presented.

Keywords: One-dimensional non-linear non-stationary of Boltzmann equation; System of Boltzmann moment equations; Microscopic maxwell boundary condition; Macroscopic boundary conditions.

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1. Introduction

Rarefied gas dynamics problems require the solution to one or another problem for the Boltzmann equation. The description of a rarefied gas using the particle distribution function refers to the transition regime between the continuous medium and free-molecular flows, and it is a rather complicated problem. The prediction of aerodynamic characteristics of aircraft at very high speeds and at high altitudes is an important problem in aerospace engineering. It is impossible to determine the aerodynamic characteristics of aircraft at high altitudes. The interaction of gas molecules with the surfaces of real bodies

has been poorly studied. The aerodynamic characteristics of aircraft at very high speeds and at high altitudes can be determined by the methods of the rarefied gas theory.^[1] To analyze the aerodynamic characteristics of aircraft in the transient regime, the complete integro-differential Boltzmann equation with appropriate boundary conditions is used. The determination of the boundary conditions on the surfaces that are streamlined with a rarefied gas is one of the most important questions in the kinetic theory of gases. In high-altitude aerodynamics, the interaction of gas with the surface of a streamlined body plays an important role.^[2] The role of the laws of interaction of molecules with the surface is more pronounced, the more rarefied the gas. The boundary conditions for the Boltzmann equation are the conditions connecting the distribution function of incident and reflected molecules. The aerothermodynamic characteristics of an

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aircraft are determined by the collisions of molecules of the oncoming gas flow with the surface of the aircraft without taking into account intermolecular collisions.

The aerothermodynamic characteristics of bodies to the gas flow are determined by the transfer of momentum and energy to the surface of the body, that is, the connection between the velocities and the energies of the molecules incident on the surface and the molecules reflected from it, which is the essence of the kinetic boundary conditions on the surface. Maxwell's boundary condition for solving specific problems more accurately describes the interaction of gas molecules with the surface.

There are several approaches to modeling rarefied flows: numerical solution of the Boltzmann equation directly, direct statistical modeling (DSM), numerical solution of model kinetic equations and the use of moment equations. Each of the listed approaches has its own advantages and disadvantages. The Boltzmann equation is the main and most general mathematical model of rarefied gas dynamics and acts as a kind of gold standard in this list. However, the Boltzmann equation is a complex nonlinear integral-differential equation. On the right-hand side, there is a five-dimensional integral that must be calculated for each point in space and each velocity at each time step. Thus, using the Boltzmann equation for numerical modeling of rarefied flows is difficult. The DSMC method is most widely used in rarefied gas dynamics Direct Simulation Monte-Carlo (DSMC),^[3] allowing for much lower costs to perform numerical modeling of stationary flows. However, DSMC has difficulties with modeling slow and unsteady flows. Model kinetic and moment equations are free from the disadvantages of the first two approaches. Moment equations are a type of continuous medium equations and describe the state of a rarefied gas in a transient regime. Model equations are kinetic nonlinear equations and are similar to the Boltzmann equation, from which they differ by a much simpler collision term. In addition, the Chapman-Enskog method is used to construct an approximate solution to the Boltzmann equation. The main assumption underlying the Chapman-Enskog method and its modifications is that the time between collisions of molecules is much shorter than all other characteristic time scales under consideration and, therefore, the gas at each point is in a state very close to the equilibrium position.^[4] The small parameter in the Chapman-Enskog theory is equal to the ratio of the mean free path to the characteristic macroscopic size. The ratio of the mean free path to the characteristic macroscopic size is usually called the Knudsen number. For the Chapman-Enskog theory to be applicable, the Knudsen number must be small. The advantage of the Chapman-Enskog method is that it leads directly to the Navier-Stokes-Fourier equations for a viscous compressible fluid. However, this method leads to higher-order equations whose status is unclear and whose practical value is negligible. The moment method is applicable to constructing an approximate solution to the Boltzmann equation for any

values of the Knudsen number. The moment method is often used to solve the initial-boundary value and initial-value problems for the Boltzmann equation.

With the help of the moment method, it is possible to determine the aerodynamic characteristics of aircraft, such as atmospheric parameters, flight speed, geometric parameters, *etc.* It should be noted that two new models of boundary conditions were proposed: diffusive-moment and specular-moment, generalizing the known boundary conditions of Cherkhinyani,^[5] and aerodynamic characteristics of space vehicles were studied by the method of direct static modeling (Monte Carlo method) and various model of the interaction of gas molecules with the surface and their effect on aerodynamic characteristic.^[6]

The physiological boundary conditions are considered at the inlets and outlets of the arterial network in terms of the lattice Boltzmann variables.^[7] The following boundary conditions: for pressure and blood flow at the inlet of the vascular network, boundary conditions for pressure and blood flow for the vessel bifurcations, wave reflection conditions (correspond to complete occlusion of the vessel) and wave absorption at the ends of the vessels (these conditions correspond to the passage of the wave without distortion), as well as RCR (Resistance–Capacitance–Resistance model)-type conditions, which are similar to electrical circuits and consist of two resistors (corresponding to the impedance of the vessel, at the end of which the boundary conditions are set and the friction forces in microcirculatory bed) and one capacitor (describing the elastic properties of arterioles). The numerical simulations were performed: the propagation of blood in a network of three vessels was considered, the boundary conditions for the blood flow were set at the entrance of the network, RCR boundary conditions were stated at the ends of the network. The solutions to lattice Boltzmann model are compared with the benchmark solutions (based on numerical calculations for second-order McCormack difference scheme without viscous terms), it is shown that the both approaches give very similar results. An approach to justifying the choice of boundary conditions on a solid surface for flows of varying degrees of nonequilibrium is proposed.^[8] Various options for setting boundary conditions are considered: no-slip boundary conditions, boundary conditions for boundary layer sliding, as well as an explicit description of the intermolecular interaction of gas molecules with molecules of a solid surface by specifying the distribution function of molecules by velocities. Numerical experiments are carried out for a plane Couette flow.

Moment methods are different from each other as sets of various systems of basic functions. For example, G. Grad derived moment system through decomposition of particles distribution function by Hermite polynomials near the local Maxwell's distributions.^[9,10] Grad used Cartesian coordinates of velocities and Grad's moment system contained unknown hydrodynamic characteristics as density, temperature, average speed, *etc.* The moment system which differs from the system

of equations of Grad was obtained.^[11] In this case, spherical coordinates of velocity and distribution function used is decomposed into a series by the eigenfunctions of the linearized collision operator,^[1,12] which is the product of the Sonin polynomials and spherical functions. The expansion coefficients, the moments of the distribution function are defined differently from Grad. The resulting system of equations corresponding to the partial sum of the series, which are called the Boltzmann's moment system equations, is a non-linear hyperbolic system relative to the moments of the particles distribution function. Differential part of the resulting system is linear and quadratic non-linearity is shaped as moments of the distribution function. Quadratic forms – the moments of the non-linear collision integrals – are calculated,^[13] and expressed in terms of Talmi,^[14] and Klebsh-Gordon coefficients.^[15] A comparison of the Grad system of equations and the Boltzmann system of moment equations is given at the end of this article.

Moment systems for the spatially homogeneous Boltzmann equation and the conditions for the representability of the solution to the spatially homogeneous Boltzmann equation in the form of the Poincare series were obtained.^[17,18] The application of the Fourier transform with respect to the velocity variable in the isotropic case greatly simplified the collision integral, and hence the calculation of moments from the collision integral.^[17] The result for the anisotropic scattering is generalized.^[18]

A systematic non-perturbative derivation of a closed system moment equations hierarchy corresponding to any classical theory is presented.^[19] The first member of the hierarchy is Euler system, which is based on Maxwellian velocity distributions, while the second one is based on non-isotropic Gaussian velocity distributions. The closure procedure has two steps: the first step ensures that every member of hierarchy is hyperbolic, has an entropy and possesses realizability of its predicted moments, and the second step involves modification of the collisional terms that is a non-linear generalization of the "diagonal approximation" of Grad, which ensures that members of the hierarchy outside the Gaussian closure will restore the correct Navier-Stokes behavior. The work of C.D. Levermore is a fundamental work for cases when closed systems of moment equations describe a transition regime.^[19]

Boltzmann equation is equivalent to an infinite system of differential equations relative to the moments of the particle distribution function in the complete system of eigenfunctions of linearized operator. As a rule, we have limited our study to finite system of equations, since the solution to an infinite system of equations is not possible. Finite moment system of equations for a specific task with a certain degree of accuracy replaces the Boltzmann equation. It is necessary to replace the boundary conditions for the particle distribution function with a series of macroscopic conditions for the moments, *i.e.*, a problem about the boundary conditions for a finite system of equations approximating the microscopic boundary conditions

for the Boltzmann equation is raised. The question of boundary conditions for a finite moment system of equations can be divided into two parts: how many conditions must be imposed and how they should be prepared. From microscopic boundary conditions for the Boltzmann equation, an infinite set of boundary conditions for each type of decomposition is obtained. However, the number of boundary conditions is determined not by the number of moment equations, *i.e.*, it is impossible, for example, to take as many boundary conditions as equations, although the number of moment equations affects the number of boundary conditions. In addition, the boundary conditions must be consistent with the moment equations and the resulting problem must be correct.

Grad described the construction of an infinite sequence of boundary conditions without matching the order of approximation of the decomposition of the boundary conditions and the decomposition of the Boltzmann equation.^[9] Boundary conditions, even one-dimensional system of Grad moment equations, are very difficult, because this system of equations is hyperbolic and contains such unknown parameters as density, temperature, average speed, *etc.* as coefficients. In this case, the characteristic equation also depends on the unknown parameters and therefore it is very difficult to formulate boundary conditions for the moment system. The boundary conditions for the 13-moment Grad system are discussed.^[20]

A new computational algorithm is proposed that will become an integral part of the moment method for solving the Boltzmann equation.^[21] The consideration is based on the principle of collision integral invariance with respect to the choice of the basic system of functions, according to which the expansion of the distribution function is carried out. The relationships between the matrix interaction elements are systematically studied in detail. For the axisymmetric case, recurrent relationships between the matrix elements are derived.

Approximation of the Boltzmann equation based on the moment method was shown.^[22] The authors propose some generalization for the formulation of the problem of moment closure from relative entropy to $\phi\phi$ -divergences and the corresponding closure based on the minimization of $\phi\phi$ -divergences. The proposed description includes the classical Grad closure based on expansion in Hermite polynomials and the Levermore closure based on entropy. It is found that the generalization of divergence-based closure allows constructing extended thermodynamic theories that avoid significant limitations of standard moment closure formulations, such as the inadmissibility of the approximate phase space distribution, the potential loss of hyperbolicity, and the singularity of flux functions at local equilibrium. The divergence-based closure leads to a hierarchy of tractable symmetric hyperbolic systems that preserve the fundamental structural properties of Boltzmann equation.

The development of continuum models for describing gases is discussed, in which particle collisions are unable to

maintain thermal equilibrium.^[23] Such situations typically arise in rarefied or diluted gases, for flows in microscopic conditions, or in general when the Knudsen number becomes significant. Continuum models in the kinetic theory of gases are based on the stochastic description of the gas using the Boltzmann equation. Extended fluid dynamics equations can be derived using moment approximations such as the regularized 13-moment equations. Moment equations are described in detail, and typical results are discussed for flow in a channel, in a cavity, and past a sphere under low Mach number conditions. Both the evolution equations and the boundary conditions are well known. In contrast, non-linear high-speed processes require special closures that are still under development. Modern approaches are discussed, as well as the problem of calculating shock wave profiles based on continuum equations.

The convergence of stable Hermite approximations (stable Hermite approximations obtained by the method of setting stable boundary conditions for Hermite approximations of arbitrary order in the case of linear kinetic equations) is studied and explicit convergence rates under suitable regularity assumptions on the exact solution are proven.^[24] The convergence rates presented are confirmed through numerical experiments involving the linearized BGK equation of rarefied gas dynamics.

A globally hyperbolic regularization was proposed to the general Grad's moment system in multidimensional spaces.^[25] Systems with moments up to an arbitrary order are studied. The characteristic rates of the regularized moment system can be analytically given and depend only on the macroscopic velocity and the temperature. The structure of the eigenvalues and eigenvectors of the coefficient matrix are fully clarified. In addition, all characteristic waves are proven to be genuinely non-linear or linearly degenerate, and the studies on the properties of rarefaction waves, contact discontinuities, and shock waves are included.

In the last two decades, the maximum entropy principle (MEP) has been successfully employed to construct macroscopic models that describe the charge and heat transport in semiconductor devices. These models are obtained, starting from the Boltzmann transport equations, for the charge and the phonon distribution functions, by taking, as macroscopic variables, suitable moments of the distributions and exploiting MEP to close the evolution equations for the chosen moments. Significant results have also been obtained for the description of charge transport in devices made of both elemental and compound semiconductors, in cases where charge confinement is present, and the carrier flow is two-dimensional or one-dimensional.^[26]

Various mathematical theories and simulation methods were developed in the past to describe gas flows in non-equilibrium, in particular, in the hypersonic rarefied mode. These methods range from the mesoscale models like Boltzmann equations and the DSMC to the high order hydrodynamic equations. The moment equations can be

derived by introducing the statistical averages in velocity space and then combining them with the Boltzmann kinetic equation. On the basis of Eu's generalized hydrodynamics and the balanced closure that was recently developed by Myong, the second-order constitutive model of Boltzmann equations that is applicable for numerical simulation of hypersonic rarefied flows is presented.^[27] Multidimensional computational models of the second-order constitutive equations are also developed based on the concept of decomposition and method of iterations. Finally, some practical applications of the second-order constitutive model to hypersonic rarefied flows like reentry vehicles with complicated geometry are described.

Approximation of the homogeneous boundary condition for the particle distribution function was shown and the correctness of the initial boundary value problem for non-stationary non-linear system of Boltzmann moment equations in a three-dimensional space was proven.^[11] More precisely, the existence of a unique generalized solution to the initial boundary value problem for the system of Boltzmann moment equations in the space of functions that are continuous in time and square-summable over the spatial variable was proven. In addition, the same study showed the approximation of the microscopic boundary conditions for Boltzmann equations. The boundary condition can be formulated as follows: to determine the specular half of the distribution function from the known one, corresponding to the falling particles. The boundary condition is specified as an integral relation between particles falling to the boundary and particles reflected from the boundary (assuming that the probability of an event is known where a particle falling on a boundary with velocity v_i is reflected with velocity v_r).

The application of Boltzmann equations is devoted to determine the aerodynamic characteristic of aircrafts.^[28-33] To analyze the aerodynamic characteristics of an aircraft in the transient mode, the complete integro-differential Boltzmann equation is used, including the term depending on the speed of the aircraft. In this case, the condition on the moving boundary must contain a parameter that depends on the surface temperature of the aircraft. In the above-mentioned works, the derivation of a new system of moment equations and Maxwell boundary condition approximation are given. The new system of equations depends on the velocity of motion and the temperature of the aircraft surface, and the macroscopic boundary conditions depend on the temperature of the aircraft surface. This system of equations differs from the Grad system of equations^[9] and Boltzmann system of moment equations,^[11] and is called the Auzhani system of moment equations. The correctness of the initial boundary value problem for the Auzhani system of moment equations under macroscopic boundary conditions in the space of functions continuous in time and square-summable over the spatial variable was proven. In addition, the aerodynamic characteristics of aircraft, such as flight speed and surface temperature, as well as atmospheric parameters, are determined. A new two-

dimensional system of moment equations is derived that takes into account the speed of motion and the surface temperature of a body immersed in a liquid.^[29] The microscopic Maxwell boundary condition for the particle distribution function is approximated, taking into account the surface temperature of a body immersed in a liquid. The issues of existence and uniqueness of a solution to the initial-boundary value problem for a nonlinear two-dimensional non-stationary system of moment equations under macroscopic boundary conditions arising from the approximation of the microscopic Maxwell boundary condition for the particle distribution function are studied.

In this paper, an approximation of microscopic boundary conditions is presented, when some molecules are reflected from the boundary specularly, and some - diffusely with a Maxwell distribution. In this case, the macroscopic boundary conditions for the system of Boltzmann moment equations are obtained from Maxwell's microscopic boundary conditions. The number of macroscopic boundary conditions at the ends of the interval is six. First, we show the correctness of the new initial boundary value problem for the system of Boltzmann moment equations. We will prove the existence and uniqueness of solutions of the initial boundary value problem for the fifth approximation of one-dimensional system of Boltzmann moment equations with macroscopic boundary conditions in space of functions continuous in time and square-summable over the spatial variable. Then, to construct an approximate solution to the initial boundary value problem for the fifth approximation of the one-dimensional system of Boltzmann moment equations, we apply the finite difference method. The differential problem is approximated by a finite-difference scheme, and an algorithm and a program for numerical implementation on a computer are compiled. At the end of the article, graphs of the moments of the particle distribution function are given.

2. Formulation of the initial boundary value problem for a one-dimensional non-linear non-stationary system of Boltzmann moment equations in fifth approximation

In the case of gas flow within a region bounded by a closed or unclosed surface, or close to a solid, the boundary conditions are specified as the ratio between the particles incident on the boundary and reflected from it. If the initial distribution of gas molecules is known, then further evolution of the gas is described by the Boltzmann integro-differential equation. Thus, the problem is reduced to solving the initial boundary value problem for the Boltzmann equation. In this section, we present the formulation of the initial boundary value problem for the one-dimensional non-stationary Boltzmann equation under Maxwell's boundary conditions, without specifying the interaction of gas with the wall. We will approximate the initial boundary value problem for the Boltzmann equation with the corresponding problem for the system of Boltzmann moment equations in the fifth approximation.

Let us present the derivation of macroscopic boundary

conditions, which are obtained from the Maxwell's microscopic boundary conditions, and the formulation of the initial boundary value problem for the system of Boltzmann moment equations. Note that the theorem on the existence of global in time solutions of initial boundary value problem for the 3-dimensional non-linear Boltzmann equation with the boundary conditions of Maxwell is proved.^[34]

Statement of the problem: Find solution to initial boundary value problem for a one-dimensional Boltzmann equation is solved by Eqs. (1)-(3):

$$\frac{\partial F}{\partial t} + |v| \cos \theta \frac{\partial F}{\partial t} = I(F, F), t \in (0, T], x \in (-a, a), v \in R_3^v \quad (1)$$

satisfying initial condition

$$F|_{t=0} = F^0(x, v), (x, v) \in [-a, a] \times R_3^v \quad (2)$$

and boundary condition

$$F^+(t, x, v_1, v_2, v_3) = \beta F^-(t, x, v_1, v_2, -v_3) + (1 - \beta) \exp\left(-\frac{|v|^2}{2R\theta}\right), v_3 = |v| \cos \theta, (n, v) = (n, |v| \cos \theta) > 0, x = -a \text{ or } x = a \quad (3)$$

where $F \equiv F(t, x, v)$ - particle distribution function in space of velocity and time; $F^0(x, v)$ - particle distribution at the initial time (fixed function); $I(F, F) \equiv \int [F(v')F(\rho') - F(v)F(\rho)] \sigma(\cos \chi) d\rho dn$ - non-linear collision operator, recorded for Maxwell molecules, n - unit outer normal vector of the boundary, χ - angle between relative velocities of the particles before and after collision as shown in Fig. 1, and $J(F, F)$ - function of t, x, v . \vec{n} being the unit vector of the relative velocity after collision.

According to boundary Eq. (3), some part of particles falling to the boundary $F^+(t, x, v_1, v_2, v_3)$ is reflected specular $F^-(t, x, v_1, v_2, -v_3)$, and other particles are absorbed into the wall and emits with Maxwell distribution, which corresponds to wall temperature θ . For parameter $\beta \in (0,1)$, $\beta = 1$ corresponds to pure specular reflection from the wall. Eq. (3) refers to the case of a wall at rest; otherwise v must be replaced by $v - u_0, u_0$ being the velocity of the wall. β, θ, u_0 may vary from point to point and over time.^[12]

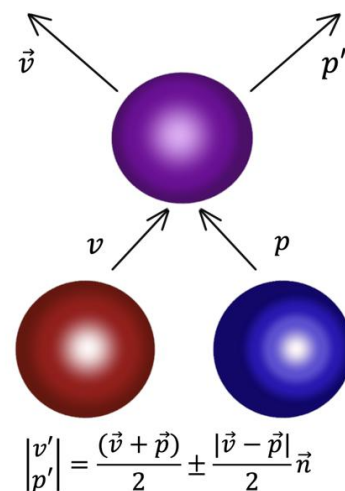


Fig. 1: Particle velocities before and after collision.

Boltzmann equation is a complex integro-differential equation. Therefore, it is impossible to find an exact solution to the problem Eqs. (1)-(3). To find an approximate solution to the problem Eqs. (1)-(3), various methods are used, for example, moment method, numerical methods, Direct Simulation Monte Carlo, etc.

To find an approximate solution to the problem Eqs. (1)-(3), we apply the moment method. For one-dimensional problems, the eigenfunctions of linearized operator are presented as follows in Eq. (4):^[1,12]

$$g_{nl}(\alpha v) = \left(\frac{\sqrt{\pi}n!(2l+1)}{2\Gamma(n+l+3/2)}\right)^2 \left(\frac{\alpha|v|}{\sqrt{2}}\right)^l S_n^{l+1/2} \left(\frac{\alpha^2|v|^2}{2}\right) P_l(\cos \theta),$$

$$2n + l = 0, 1, 2, \dots \tag{4}$$

where $S_n^{l+1/2} \left(\frac{\alpha^2|v|^2}{2}\right)$ - Sonin polynomials, $P_l(\cos \theta)$ - Legendre polynomials, Γ - Gamma function, $\alpha^2 = 1/(R\theta)$, R - Boltzmann constant.

The approximate solution to the one-dimensional problem Eqs. (1)-(3) is defined as follows Eqs. (5)-(8):

$$F_5(t, x, v) = F_0(\alpha|v|) \sum_{2n+l=0}^5 F_{nl}(t, x) g_{nl}(\alpha v) \tag{5}$$

$$\int_{R^3} \left(\frac{\partial F_5}{\partial t} + |v| \cos \theta \frac{\partial F_5}{\partial x} - J(F_5, F_5)\right) g_{nl}(\alpha v) dv = 0, 2n + l = 0, 1, \dots, 5, (t, x) \in (0, T] \times (-a, a) \tag{6}$$

$$\int_{R^3} [F_5(0, x, v) - F_5^0(x, v)] g_{nl}(\alpha v) dv = 0, 2n + l = 0, 1, \dots, 5, x \in (-a, a) \tag{7}$$

$$\int_{(n,v)>0} (n, v) (t, x, v) g_{n,2l}(\alpha v) - \beta \int_{(n,v)<0} (n, v) (t, x, v) g_{n,2l}(\alpha v) dv - (1 - \beta) \int_{(n,v)<0} (n, v) \exp\left(-\frac{|v|^2}{2RT_0}\right) g_{n,2l}(\alpha v) dv = 0$$

$$2(n + l) = 0, 2, 4, x = -\alpha \text{ or } x = -\alpha \tag{8}$$

where $n = (0, 0, 1)$ with $x = \alpha$ and $n = (0, 0, -1)$ with $x = -\alpha$, $F_0(\alpha|v|)$ - global Maxwell distribution; $F_0(\alpha|v|) = \left(\frac{\alpha^2}{2\pi}\right)^{3/2} \cdot e^{-\frac{\alpha^2|v|^2}{2}}$; $F_5^0(x, v) = F_0(\alpha|v|) \sum_{2n+l=0}^5 F_{nl}^0(x) g_{nl}(\alpha v)$, $F_{nl}^0(x) = \int_{R^3} F_5^0(x, v) g_{nl}(\alpha v) dv$,

$$F_{nl}(t, x) = \int_{R^3} F_5(t, x, v) g_{nl}(\alpha v) dv$$

In the general case, the approximation of the boundary condition Eq. (3) depends on the parity or oddness of the approximation of the system of Boltzmann moment equations.^[28] When approximating a microscopic boundary condition, we took into account the approximation of the Boltzmann equation by moment equations corresponding to the fifth approximation. Thus, the approximation orders for the expansion of the boundary condition and the expansion of the Boltzmann equation are consistent. Macroscopic conditions Eq. (8) were called Maxwell – Auzhan boundary conditions.^[27] System of Boltzmann moment Eq. (6),

corresponding to the decomposition Eq. (5), can be written in the expanded form:^[36]

$$\frac{\partial F_{nl}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left[l \left(\sqrt{\frac{2(n+l+1/2)}{(2l-1)(2l+1)}} F_{n,l-1} - \sqrt{\frac{2(n+1)}{(2l-1)(2l+1)}} F_{n+1,l-1} \right) + (l+1) \left(\sqrt{\frac{2(n+l+3/2)}{(2l+1)(2l+3)}} F_{n,l+1} - \sqrt{\frac{2n}{(2l+1)(2l+3)}} F_{n-1,l+1} \right) \right] = I_{nl}, 2n + l = 0, 1, \dots, 5, \tag{9}$$

where the moments of the collision integral can be expressed in terms of coefficients Talmi and Klebsh-Gordon as follows:^[13-16]

Generalized Talmi coefficients (Eq. (10))

$$I_{nl} = \sum \langle N_3 L_3 n_3 l_3 : l | nl 0 0 : l \rangle \langle N_3 L_3 n_3 l_3 : l | n_1 l_1 n_2 l_2 : l \rangle \langle l_1 0 l_2 0 / l 0 \rangle (\sigma_{l_3} - \sigma_0) f_{n_1 l_1} f_{n_2 l_2}, \langle N_3 L_3 n_3 l_3 : l | n_1 l_1 n_2 l_2 : l \rangle \tag{10}$$

where $(l_1 0 l_2 0 / l 0)$ - Klebsh-Gordon coefficients.

Let us present the fifth approximation of the system of Boltzmann moment equations in expanded form. If in Eq. (9), $2n + l$ is from 0 to 5, we get the following system of equations (Eqs. (11)-(23)):

$$\frac{\partial F_{00}}{\partial t} + \frac{1}{\alpha} \frac{\partial F_{01}}{\partial x} = 0 \tag{11}$$

$$\frac{\partial F_{02}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\frac{2}{\sqrt{3}} F_{01} + \frac{3}{\sqrt{5}} F_{03} - \frac{2\sqrt{2}}{\sqrt{15}} F_{11} \right) = I_{02} \tag{12}$$

$$\frac{\partial F_{04}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(4 \left(\frac{1}{\sqrt{7}} F_{03} - \sqrt{\frac{2}{7.9}} F_{13} \right) + \frac{5}{3} F_{05} \right) = I_{04} \tag{13}$$

$$\frac{\partial F_{10}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(-\sqrt{\frac{2}{3}} F_{01} + \sqrt{\frac{5}{3}} F_{11} \right) = 0 \tag{14}$$

$$\frac{\partial F_{12}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(2 \left(\sqrt{\frac{7}{3.5}} F_{11} - \frac{2}{\sqrt{3.5}} F_{21} \right) + 3 \left(\frac{3}{\sqrt{5.7}} F_{13} - \sqrt{\frac{2}{5.7}} F_{03} \right) \right) = I_{12} \tag{15}$$

$$\frac{\partial F_{12}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(2 \left(\sqrt{\frac{7}{3.5}} F_{11} - \frac{2}{\sqrt{3.5}} F_{21} \right) + 3 \left(\frac{3}{\sqrt{5.7}} F_{13} - \sqrt{\frac{2}{5.7}} F_{03} \right) \right) = I_{12} \tag{16}$$

$$\frac{\partial F_{20}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\sqrt{\frac{7}{3}} F_{21} - \frac{2}{\sqrt{3}} F_{11} \right) = I_{20} \tag{17}$$

$$\frac{\partial F_{01}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(F_{00} + \frac{2}{\sqrt{3}} F_{02} - \sqrt{\frac{2}{3}} F_{10} \right) = 0 \tag{18}$$

$$\frac{\partial F_{03}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left[3 \left(\frac{1}{\sqrt{5}} F_{02} - \sqrt{\frac{2}{5.7}} F_{12} \right) + \frac{4}{\sqrt{7}} F_{04} \right] = I_{03} \tag{19}$$

$$\frac{\partial F_{05}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\frac{5}{3} F_{04} \right) = I_{05} \tag{20}$$

$$\frac{\partial F_{11}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\sqrt{\frac{5}{3}} F_{10} - \frac{2}{\sqrt{3}} F_{20} + 2 \left(\sqrt{\frac{7}{3.5}} F_{12} - \sqrt{\frac{2}{3.5}} F_{02} \right) \right) = I_{11} \tag{21}$$

$$\frac{\partial F_{13}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\frac{9}{\sqrt{5 \cdot 7}} F_{12} - \frac{4}{3} \sqrt{\frac{2}{7}} F_{04} \right) = I_{13} \quad (22)$$

$$\frac{\partial F_{21}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\frac{7}{3} F_{20} - \frac{2}{\sqrt{3 \cdot 5}} F_{12} \right) = I_{21}, x \in (-a, a), t > 0 \quad (23)$$

where $F_{00} = F_{00}(t, x), F_{02} = F_{02}(t, x), \dots, F_{21} = F_{21}(t, x)$ are the moments of particles distribution function; $I_{02}, I_{04}, I_{12}, I_{20}, I_{03}, I_{05}, I_{11}, I_{13}, I_{21}$ are the moments of collision integral. Moments of collision integral are non-sign defined square forms. To calculate the moments of the collision integral, we use a table and a program.^[15,16]

$$I_{02} = (\sigma_2 - \sigma_0)(f_{00}f_{02} - f_{01}^2/\sqrt{3})/2 \quad (24)$$

$$I_{03} = \frac{1}{4}(\sigma_3 + 3\sigma_1 - 4\sigma_0)f_{00}f_{03} + \frac{1}{4\sqrt{5}}(2\sigma_1 + \sigma_0 - 3\sigma_3)f_{01}f_{02} \quad (25)$$

$$I_{11} = (\sigma_1 - \sigma_0) \left(f_{00}f_{11} + \frac{1}{2} \sqrt{\frac{5}{3}} f_{10}f_{01} - \frac{\sqrt{2}}{\sqrt{15}} f_{01}f_{02} \right) \quad (26)$$

$$I_{04} = -\frac{1}{2} \sqrt{\frac{3}{2}} (\sigma_1 - \sigma_0) f_{00}f_{04} - \frac{1}{8} (\sigma_4 - \sigma_0) \left(f_{00}f_{04} - \frac{2}{\sqrt{7}} f_{01}f_{03} \right) \quad (27)$$

$$I_{12} = \frac{3}{70\sqrt{10}} \frac{\sigma_2 - \sigma_0}{6} \left(-\frac{2\sqrt{2}}{21} f_{02}^2 + f_{00}f_{12} \right) \quad (28)$$

$$I_{20} = \frac{\sigma_1 - \sigma_0}{2} \left(f_{20}f_{00} + \frac{1}{\sqrt{3}} f_{11}f_{01} \right) + \frac{2(\sigma_2 - \sigma_0)}{\sqrt{6}} \left(\frac{1}{\sqrt{6}} f_{20}f_{00} - \frac{\sqrt{5}}{3} f_{10}^2 + \frac{1}{3\sqrt{5}} f_{20}^2 \right) \quad (29)$$

$$I_{13} = \frac{\sigma_1 - \sigma_0}{2\sqrt{7}} \left(0,79\sqrt{3}f_{05}f_{00} + 0,79\sqrt{3}f_{04}f_{01} + \frac{1}{\sqrt{5 \cdot 70}} f_{02}f_{03} \right) - \frac{\sigma_2 - \sigma_0}{2\sqrt{5}} \left(\frac{3}{\sqrt{5 \cdot 70}} f_{04}f_{01} + \frac{14\sqrt{3}}{5\sqrt{2}} f_{02}f_{03} \right) + \frac{\sigma_3 - \sigma_0}{\sqrt{51}} \left(\frac{1}{\sqrt{70 \cdot 5}} f_{04}f_{01} - \frac{7}{5\sqrt{3 \cdot 15}} f_{02}f_{03} \right) - \frac{\sigma_4 - \sigma_0}{2\sqrt{7}} \left(0,79\sqrt{3}f_{05}f_{00} - \frac{2,37}{\sqrt{5}} f_{04}f_{01} + \frac{2\sqrt{3}}{\sqrt{70 \cdot 15}} f_{02}f_{03} \right) \quad (30)$$

$$I_{05} = (\sigma_2 - \sigma_0) \left(\frac{5}{8} f_{00}f_{05} + \frac{5}{24} f_{01}f_{04} + \frac{1}{2\sqrt{6}} f_{03}f_{02} \right) - (\sigma_3 - \sigma_0) \left(\frac{5}{8} f_{00}f_{05} + \frac{5}{24} f_{01}f_{04} + \frac{1}{2\sqrt{6}} f_{03}f_{02} \right) + (\sigma_1 - \sigma_0) \left(0,063f_{00}f_{05} + \frac{0,14\sqrt{5}}{3} f_{01}f_{04} + \frac{0,177}{\sqrt{15}} f_{03}f_{02} \right) - (\sigma_4 - \sigma_0) \left(0,063f_{00}f_{05} + \frac{0,14\sqrt{5}}{3} f_{01}f_{04} + \frac{0,177}{\sqrt{15}} f_{03}f_{02} \right) \quad (31)$$

$$I_{21} = (\sigma_1 - \sigma_0) \left(-0,154f_{00}f_{05} + \frac{0,62\sqrt{5}}{3} f_{01}f_{04} + \frac{0,6}{\sqrt{35}} f_{03}f_{02} \right) - (\sigma_2 - \sigma_0) \left(\frac{0,56\sqrt{5}}{2} f_{00}f_{05} + \frac{\sqrt{5}}{6} f_{01}f_{04} + \frac{3}{4\sqrt{35}} f_{03}f_{02} \right) + (\sigma_5 - \sigma_0) \left(0,14f_{00}f_{05} + 0,14f_{01}f_{04} + \frac{0,531}{2\sqrt{7}} f_{03}f_{02} \right) \quad (32)$$

The system of Eqs. (24-32) contains three homogenous equations corresponding to the law of conservation of mass (Eq. (1)), the law of conservation of momentum (Eq. (8)), and the law of conservation of energy (Eq. (5)). We present these equations separately in Eqs. (33-35):

$$\frac{\partial F_{00}}{\partial t} + \frac{1}{\alpha} \frac{\partial F_{01}}{\partial x} = 0 \quad (33)$$

$$\frac{\partial F_{01}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(F_{00} + \frac{2}{\sqrt{3}} F_{02} - \sqrt{\frac{2}{3}} F_{10} \right) = 0 \quad (34)$$

$$\frac{\partial F_{10}}{\partial t} + \frac{1}{\alpha} \frac{\partial}{\partial x} \left(-\sqrt{\frac{2}{3}} F_{01} + \sqrt{\frac{5}{3}} F_{11} \right) = 0 \quad (35)$$

Let us introduce hydrodynamic quantities into consideration (mass $m = 1$) in Eqs. (36) and (37):

$$\rho = n = \int F dv, V = \frac{1}{n} \int v F dv, P_{ij} = \int C_i C_j F dv \quad (36)$$

$$q_i = \frac{1}{2} \int C^2 C_i F dv, \frac{3}{2} R\theta = \frac{1}{n} \int \frac{1}{2} C^2 F dv \quad (37)$$

where n - number of particles per unit volume of gas, ρ - density of gas, V - average velocity of gas, P_{ij} - stress tensor, q_i - components of heat flow, θ - temperature of gas, R - Boltzmann's constant, $C = v - V$ - thermal or own velocity of gas. The moments of the particle distribution function can be expressed in terms of hydrodynamic characteristics:

$$F_{00} = \int F g_{00} dv = \int F dv = \rho; \quad F_{01} = \int F g_{01} dv =$$

$$\int F dv \cos \theta dv = \alpha V \rho; \quad F_{02} = \int F g_{02} dv = \frac{\alpha^2}{2\sqrt{3}} (P_{33} + \rho V^2);$$

$$F_{03} = \int F g_{03} dv = \frac{\alpha^3}{\sqrt{15}} (2q + 9V\rho R\theta + V^3\rho); \quad F_{10} =$$

$$\int F g_{10} dv = \alpha \sqrt{\frac{5}{2}} \rho V - \sqrt{\frac{2}{5}} \alpha^3 \left(q + VP_{33} + \rho V \left(\frac{3P}{2\rho} + \frac{V^2}{2} \right) \right) \quad (38)$$

If we substitute the values of $F_{00}, F_{01}, F_{02}, F_{10}, F_{11}$ into the system of Eqs. (24) from (38), we obtain the following system of equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0, \frac{\partial \rho V}{\partial t} + \frac{\partial}{\partial x} (P_{ij} + \rho V^2) = 0 \quad (39)$$

$$\frac{\partial}{\partial t} \rho \left(\varepsilon + \frac{V^2}{2} \right) + \frac{\partial}{\partial x} \left(\rho V \left(\varepsilon + \frac{P_{33}}{\rho} + \frac{V^2}{2} \right) + q \right) = 0, \varepsilon = \frac{3}{2} R\theta \quad (40)$$

The system of Eqs. (39) and (40) is not closed, because the number of equations is three and the number of unknowns is five. To close the system of Eqs. (39) and (40), we express P_{33} and q through the moments of lower orders. If $P_{33} = p, q = 0$ (ideal fluid), we obtain the Euler' system equations (Eqs. (41-43)):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0 \quad (41)$$

$$\frac{\partial \rho V}{\partial t} + \frac{\partial}{\partial x} (p + \rho V^2) = 0 \quad (42)$$

$$\frac{\partial}{\partial t} \left[\rho \left(\varepsilon + \frac{1}{2} V^2 \right) \right] + \frac{\partial}{\partial x} \left[\rho V \left(\varepsilon + \frac{p}{\rho} + \frac{1}{2} V^2 \right) \right] = 0 \quad (43)$$

For Navier-Stokes and Fourier fluid (viscous heat-conducting fluid), we get Eq. (44):

$$P_{33} = p - \frac{4}{3} \mu \frac{\partial V}{\partial x}, q = -\kappa \frac{\partial \theta}{\partial x} \quad (44)$$

where p - pressure, μ - viscosity coefficient, κ - heat conducting coefficient. If we substitute the values of coefficients P_{33} and q from Eq. (44) into Eq. (40), we obtain the Navier-Stokes and Fourier system of equations (Eq. (45-47)):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0 \quad (45)$$

$$\rho \left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} \right) = \frac{4}{3} \mu \frac{\partial^2 V}{\partial x^2} - \frac{\partial p}{\partial x} \quad (46)$$

$$\rho \left(\frac{\partial \varepsilon}{\partial t} + V \frac{\partial \varepsilon}{\partial x} \right) = \kappa \frac{\partial^2 \theta}{\partial x^2} + \frac{4}{3} \mu \left(\frac{\partial V}{\partial x} \right)^2 - p \frac{\partial V}{\partial x}, \varepsilon = \frac{3}{2} R\theta \quad (47)$$

The number of equations in Eqs. (11)-(23) is twelve,

therefore the fifth approximation of the system of Boltzmann moment equations contains twelve for the moments of the particle distribution function. Equality Eq. (8) is an approximation of Maxwell's microscopic condition Eq. (3). Eq. (8) presents macroscopic boundary condition corresponding to the fifth approximation of the system of Boltzmann moment equations, and integration in Eq. (8) is carried out over the velocity half-space. We will write the macroscopic condition in expanded form.

At the boundary $x = -a$ or $x = a$, we will set the following boundary conditions (normal vector $n = (0,0,-1)$ with $x = -a$ and $n = (0,0,1)$ with $x = a$).

$$\begin{aligned} & \int_0^\infty \int_0^1 \int_0^{2\pi} (-|v|\mu) F_5^+(t, -a, v) g_{n,2l}(\alpha v) dv - \\ & \beta \int_0^\infty \int_{-1}^0 \int_0^{2\pi} (-|v|\mu) F_5^-(t, -a, v) g_{n,2l}(\alpha v) dv - (1 - \\ & \beta) \int_0^\infty \int_0^1 \int_0^{2\pi} (-|v|\mu) \exp\left(-\frac{|v|^2}{2R\theta}\right) g_{n,2l}(\alpha v) dv = 0 \end{aligned} \quad (48)$$

where $2(n+l) = 0, 2, 4 \Rightarrow n = 1 = 0, n = 0, l = 1; n = 1, l = 0; n = 0; l = 2; n = 1, l = 1; n = 2, l = 0$

$$\begin{aligned} & \int_0^\infty \int_0^1 \int_0^{2\pi} |v|\mu F_5^+(t, a, v) g_{n,2l}(\alpha v) dv - \\ & \beta \int_0^\infty \int_{-1}^0 \int_0^{2\pi} |v|\mu F_5^-(t, a, v) g_{n,2l}(\alpha v) dv - (1 - \\ & \beta) \int_0^\infty \int_{-1}^0 \int_0^{2\pi} |v|\mu \exp\left(-\frac{|v|^2}{2R\theta}\right) g_{n,2l}(\alpha v) dv = 0 \end{aligned} \quad (49)$$

where $2(n+l) = 0, 2, 4 \Rightarrow n = l = 0; n = 0, l = 1; n = 1, l = 0; n = 0; l = 2; n = 1, l = 1; n = 2, l = 0; \mu = \cos \theta, dv = |v|^2 d|v| d\mu d\varphi, F_5^\pm(t, x, v) = F_0(\alpha|v|) \sum_{2n+l=0}^5 F_{nl}^\pm(t, x) g_{nl}(\alpha v)$

Let's write $F_5^\pm(t, -a, v)$ in the following form:

$$\begin{aligned} F_5^\pm(t, -a, v) &= \frac{1}{2} [F_5^\pm(t, -a, v) - F_5^\pm(t, -a, -v)] + \\ & \frac{1}{2} [F_5^\pm(t, -a, v) + F_5^\pm(t, -a, -v)] \end{aligned} \quad (50)$$

Notice, that

$$F_5^\pm(t, -a, v) - F_5^\pm(t, -a, -v) = \sum_{2n+l=0}^5 F_{nl}^\pm(t, -a) [g_{nl}(\alpha v) - g_{nl}(-\alpha v)] \quad (51)$$

$$F_5^\pm(t, -a, v) + F_5^\pm(t, -a, -v) = \sum_{2n+l=0}^5 F_{nl}^\pm(t, -a) [g_{nl}(\alpha v) + g_{nl}(-\alpha v)] \quad (52)$$

$$g_{nl}(\alpha v) \pm g_{nl}(-\alpha v) = \left(\frac{\sqrt{\pi} n!(2l+1)}{2\Gamma(n+l+3/2)}\right)^{1/2} \left(\frac{\alpha|v|}{\sqrt{2}}\right)^l S_n^{l+1/2} \left(\frac{\alpha^2 v^2}{2}\right) [P_l(\mu) \pm P_l(-\mu)] \quad (53)$$

$$P_l(\mu) + P_l(-\mu) = \begin{cases} 0, & l = 2m - 1 \\ 2P_l(\mu), & l = 2m \end{cases} = \begin{cases} 0, & \text{listheodd} \\ 2P_{2l}(\mu) \end{cases} \quad (54)$$

$$P_l(\mu) - P_l(-\mu) = \begin{cases} 0, & \text{listheeven} \\ 2P_{2l+1}(\mu), & \text{listheodd} \end{cases} \quad (55)$$

Taking into account the last relations for the Legendre polynomials, the expressions $F_5^\pm(t, -a, v) + F_5^\pm(t, -a, -v)$ and $F_5^\pm(t, -a, v) - F_5^\pm(t, -a, -v)$, which can be written in the following form:

$$F_5^\pm(t, -a, v) + F_5^\pm(t, -a, -v) = 2 \sum_{2n+l=0}^5 F_{n,2l}^\pm(t, -a) g_{n,2l}(\alpha v) \quad (56)$$

$$F_5^\pm(t, -a, v) - F_5^\pm(t, -a, -v) = 2 \sum_{2n+l=0}^5 F_{n,2l+1}^\pm(t, -a) g_{n,2l+1}(\alpha v) \quad (57)$$

Then Eq.(11) can be rewritten in the following form:

$$\begin{aligned} & \int_0^\infty \int_{-1}^0 \int_0^{2\pi} (-|v|\mu) \left\{ \sum_{2n+l=0}^2 [F_{n,2l+1}^+(t, -a) g_{n,2l+1}(\alpha v) + \right. \\ & F_{n,2l}^+(t, -a) g_{n,2l}(\alpha v)] g_{k,2L}(\alpha v) dv - \\ & \beta \int_0^\infty \int_0^1 \int_0^{2\pi} (|v|\mu) \left\{ \sum_{2n+l=0}^2 [F_{n,2l+1}^-(t, -a) g_{n,2l+1}(\alpha v) + \right. \\ & F_{n,2l}^-(t, -a) g_{n,2l}(\alpha v)] g_{k,2L}(\alpha v) dv - (1 - \\ & \beta) \int_0^\infty \int_0^1 \int_0^{2\pi} \left(\frac{|v|^2}{2R\theta}\right)_{k,2L}^{(\alpha v)} \int_{-1}^1 \int_0^{2\pi} |v|\mu \exp 2(k+L) = 0, 2, 4 \end{aligned} \quad (58)$$

From Eq. (58), it follows that the number of boundary conditions at the left end of the interval $(-a, a)$ is equal to 6. In a similar way, it can be shown that the number of boundary conditions at the right end of the interval $(-a, a)$ is also equal to 6. If in Eqs. (48) and (49) instead of $g_{n,2l}(\alpha v)$, we substitute the values

$$\begin{aligned} g_{00} &= 1 \\ g_{02}(\alpha v) &= \sqrt{\frac{4}{3}} \left(\frac{\alpha|v|}{\sqrt{2}}\right)^2 \frac{1}{2} (3 \cos^2 \theta - 1) \\ g_{04}(\alpha v) &= \frac{1}{\sqrt{105}} \left(\frac{\alpha|v|}{\sqrt{2}}\right)^4 (35 \cos^4 \theta - 30 \cos^2 \theta + 3) \\ g_{10}(\alpha v) &= \sqrt{\frac{2}{3}} \left(\frac{3}{2} - \frac{\alpha^2 |v|^2}{\sqrt{2}}\right) \\ g_{12} &= \sqrt{\frac{2}{21}} \left(\frac{\alpha|v|}{\sqrt{2}}\right)^2 \left(\frac{7}{2} - \frac{\alpha^2 v^2}{2}\right) (3 \cos^2 \theta - 1) \\ g_{20} &= \sqrt{\frac{2}{15}} \left[\left(\frac{3}{2} - \frac{\alpha^2 v^2}{2}\right) \left(\frac{5}{2} - \alpha^2 v^2\right) - \frac{\alpha^2 v^2}{2}\right] \end{aligned}$$

Then after integration, we obtain the macroscopic boundary conditions for fifth approximation of the system Boltzmann moment equations^[28] (omitting cumbersome calculations of boundary integrals): at $x = -a$

$$\begin{aligned} & \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\sqrt{2} f_{00}^+ - \sqrt{\frac{2}{3}} f_{02}^+ + \frac{\sqrt{2}}{\sqrt{105}} f_{04}^+ + \frac{1}{\sqrt{3}} f_{10}^+ - \frac{1}{\sqrt{21}} f_{12}^+ + \right. \right. \\ & \left. \left. \frac{1}{2\sqrt{15}} f_{20}^+ \right) + f_{01}^+ \right]_{x=-a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\sqrt{2} f_{00}^- + \sqrt{\frac{2}{3}} f_{02}^- - \right. \right. \\ & \left. \left. \frac{\sqrt{2}}{\sqrt{105}} f_{04}^- - \frac{1}{\sqrt{3}} f_{10}^- + \frac{1}{\sqrt{21}} f_{12}^- - \frac{1}{2\sqrt{15}} f_{20}^- \right) + f_{01}^- \right]_{x=-a} - \\ & \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{4\sqrt{2}} = 0 \end{aligned} \quad (59)$$

$$\begin{aligned} & \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\sqrt{\frac{2}{3}} f_{00}^- - 2\sqrt{2} f_{02}^+ - \frac{13}{3} \sqrt{\frac{2}{35}} f_{04}^+ + f_{10}^+ + \frac{2}{\sqrt{7}} f_{12}^+ - \right. \right. \\ & \left. \left. \frac{1}{2\sqrt{5}} f_{20}^+ \right) + \frac{2}{\sqrt{3}} f_{01}^+ + \frac{3}{\sqrt{5}} f_{03}^+ - \frac{2\sqrt{2}}{\sqrt{15}} f_{11}^+ \right]_{x=-a} - \frac{\beta}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\sqrt{\frac{2}{3}} f_{00}^- + \right. \right. \\ & \left. \left. 2\sqrt{2} f_{02}^- + \frac{13}{3} \sqrt{\frac{2}{35}} f_{04}^- - f_{10}^- - \frac{2}{\sqrt{7}} f_{12}^- + \frac{1}{2\sqrt{5}} f_{20}^- \right) + \frac{2}{\sqrt{3}} f_{01}^- + \right. \\ & \left. \frac{3}{\sqrt{5}} f_{03}^- - \frac{2\sqrt{2}}{\sqrt{15}} f_{11}^- \right]_{x=-a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{8\sqrt{6}} = 0 \end{aligned} \quad (60)$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{2}{\sqrt{210}} f_{00}^+ - \frac{26}{3\sqrt{70}} f_{02}^+ + \frac{4}{7\sqrt{2}} f_{04}^+ - \frac{5}{3\sqrt{35}} f_{10}^+ + \frac{13}{\sqrt{245}} f_{12}^+ - \right. \right.$$

$$\frac{1}{2\sqrt{7}}f_{20}^+) + \frac{4}{\sqrt{7}}f_{03}^+ + \frac{5}{3}f_{05}^+ - \frac{4}{3}\sqrt{\frac{2}{7}}f_{13}^+ \Big|_{x=-a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{2}{\sqrt{210}}f_{00}^- + \frac{26}{3\sqrt{70}}f_{02}^- - \frac{4}{7\sqrt{2}}f_{04}^- + \frac{5}{3\sqrt{35}}f_{10}^- - \frac{13}{\sqrt{245}}f_{12}^- + \frac{1}{2\sqrt{7}}f_{20}^- \right) + \frac{4}{\sqrt{7}}f_{03}^- + \frac{5}{3}f_{05}^- - \frac{4}{3}\sqrt{\frac{2}{7}}f_{13}^- \right] \Big|_{x=-a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{8\sqrt{3}} = 0 \tag{61}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{3}}f_{00}^+ + f_{02}^+ - \frac{5}{3\sqrt{35}}f_{04}^+ - \frac{6}{\sqrt{2}}f_{10}^+ - \frac{3}{\sqrt{14}}f_{12}^+ + \frac{5}{2\sqrt{10}}f_{20}^+ \right) - \sqrt{\frac{2}{3}}f_{01}^- + \sqrt{\frac{5}{3}}f_{11}^- \right] \Big|_{x=-a} + \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{16\sqrt{210}} = 0 \tag{62}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{1}{\sqrt{21}}f_{00}^+ + \frac{2}{\sqrt{7}}f_{02}^+ + \frac{13}{\sqrt{245}}f_{04}^+ - \frac{3}{\sqrt{14}}f_{10}^+ - \frac{34}{7\sqrt{2}}f_{12}^+ - \frac{23}{4}\sqrt{\frac{2}{35}}f_{20}^+ \right) - 3\sqrt{\frac{2}{35}}f_{03}^- + 2\sqrt{\frac{7}{15}}f_{11}^+ + \frac{9}{\sqrt{35}}f_{13}^+ - \frac{4}{\sqrt{15}}f_{21}^+ \right] \Big|_{x=-a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{21}}f_{00}^- - \frac{2}{\sqrt{7}}f_{02}^- - \frac{13}{\sqrt{245}}f_{04}^- + \frac{3}{\sqrt{14}}f_{10}^- + \frac{34}{7\sqrt{2}}f_{12}^- + \frac{23}{4}\sqrt{\frac{2}{35}}f_{20}^- \right) - 3\sqrt{\frac{2}{35}}f_{03}^- + 2\sqrt{\frac{7}{15}}f_{11}^- + \frac{9}{\sqrt{35}}f_{13}^- - \frac{4}{\sqrt{15}}f_{21}^- \right] \Big|_{x=-a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{4\sqrt{35}} = 0 \tag{63}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{15}}f_{00}^+ - \frac{1}{2\sqrt{5}}f_{02}^+ - \frac{1}{2\sqrt{7}}f_{04}^+ + \frac{5}{2\sqrt{10}}f_{10}^+ - \frac{23\sqrt{2}}{4\sqrt{35}}f_{12}^+ - \frac{15}{\sqrt{2}}f_{20}^+ \right) - \frac{2}{\sqrt{3}}f_{11}^+ + \sqrt{\frac{7}{3}}f_{21}^+ \right] \Big|_{x=-a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{1}{2\sqrt{15}}f_{00}^- + \frac{1}{2\sqrt{5}}f_{02}^- + \frac{1}{2\sqrt{7}}f_{04}^- - \frac{5}{2\sqrt{10}}f_{10}^- + \frac{23\sqrt{2}}{4\sqrt{35}}f_{12}^- + \frac{15}{\sqrt{2}}f_{20}^- \right) - \frac{2}{\sqrt{3}}f_{11}^- + \sqrt{\frac{7}{3}}f_{21}^- \right] \Big|_{x=-a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{\sqrt{15}} = 0 \tag{64}$$

at $x = a$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\sqrt{2}f_{00}^+ - \sqrt{\frac{2}{3}}f_{02}^+ + \frac{\sqrt{2}}{\sqrt{105}}f_{04}^+ + \frac{1}{\sqrt{3}}f_{10}^+ - \frac{1}{\sqrt{21}}f_{12}^+ + \frac{1}{2\sqrt{15}}f_{20}^+ \right) - f_{01}^+ \right] \Big|_{x=a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\sqrt{2}f_{00}^- + \sqrt{\frac{2}{3}}f_{02}^- - \frac{\sqrt{2}}{\sqrt{105}}f_{04}^- - \frac{1}{\sqrt{3}}f_{10}^- + \frac{1}{\sqrt{21}}f_{12}^- - \frac{1}{2\sqrt{15}}f_{20}^- \right) - f_{01}^- \right] \Big|_{x=a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{4\sqrt{2}} = 0 \tag{65}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\sqrt{\frac{2}{3}}f_{00}^+ - 2\sqrt{2}f_{02}^+ - \frac{13}{3}\sqrt{\frac{2}{35}}f_{04}^+ + f_{10}^+ + \frac{2}{\sqrt{7}}f_{12}^+ - \frac{1}{2\sqrt{5}}f_{20}^+ \right) - \frac{2}{\sqrt{3}}f_{01}^+ - \frac{3}{\sqrt{5}}f_{03}^+ + \frac{2\sqrt{2}}{\sqrt{15}}f_{11}^+ \right] \Big|_{x=a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\sqrt{\frac{2}{3}}f_{00}^- + \frac{2}{\sqrt{3}}f_{02}^- + \frac{13}{3}\sqrt{\frac{2}{35}}f_{04}^- - f_{10}^- - \frac{2}{\sqrt{7}}f_{12}^- + \frac{1}{2\sqrt{5}}f_{20}^- \right) - \frac{2}{\sqrt{3}}f_{01}^- - \frac{3}{\sqrt{5}}f_{03}^- + \frac{2\sqrt{2}}{\sqrt{15}}f_{11}^- \right] \Big|_{x=a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{8\sqrt{6}} = 0 \tag{66}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{2}{\sqrt{210}}f_{00}^+ - \frac{26}{3\sqrt{70}}f_{02}^+ + \frac{4}{7\sqrt{2}}f_{04}^+ - \frac{5}{3\sqrt{35}}f_{10}^+ + \frac{13}{\sqrt{245}}f_{12}^+ - \frac{1}{2\sqrt{7}}f_{20}^+ \right) - \frac{4}{\sqrt{7}}f_{03}^+ - \frac{5}{3}f_{05}^+ + \frac{4}{3}\sqrt{\frac{2}{7}}f_{13}^+ \right] \Big|_{x=a} -$$

$$\beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{2}{\sqrt{210}}f_{00}^- + \frac{26}{3\sqrt{70}}f_{02}^- - \frac{4}{7\sqrt{2}}f_{04}^- + \frac{5}{3\sqrt{35}}f_{10}^- - \frac{13}{\sqrt{245}}f_{12}^- + \frac{1}{2\sqrt{7}}f_{20}^- \right) - \frac{4}{\sqrt{7}}f_{03}^- - \frac{5}{3}f_{05}^- + \frac{4}{3}\sqrt{\frac{2}{7}}f_{13}^- \right] \Big|_{x=a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{8\sqrt{3}} = 0 \tag{67}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{3}}f_{00}^+ + f_{02}^+ - \frac{5}{3\sqrt{35}}f_{04}^+ - \frac{6}{\sqrt{2}}f_{10}^+ - \frac{3}{\sqrt{14}}f_{12}^+ + \frac{5}{2\sqrt{10}}f_{20}^+ \right) + \sqrt{\frac{2}{3}}f_{01}^+ - \sqrt{\frac{5}{3}}f_{11}^+ \right] \Big|_{x=a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{1}{\sqrt{3}}f_{00}^- - f_{02}^- + \frac{5}{3\sqrt{35}}f_{04}^- + \frac{6}{\sqrt{2}}f_{10}^- + \frac{3}{\sqrt{14}}f_{12}^- - \frac{5}{2\sqrt{10}}f_{20}^- \right) + \sqrt{\frac{2}{3}}f_{01}^- - \sqrt{\frac{5}{3}}f_{11}^- \right] \Big|_{x=a} + \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{16\sqrt{210}} = 0 \tag{68}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{1}{\sqrt{21}}f_{00}^+ + \frac{2}{\sqrt{7}}f_{02}^+ + \frac{13}{\sqrt{245}}f_{04}^+ - \frac{3}{\sqrt{14}}f_{10}^+ - \frac{34}{7\sqrt{2}}f_{12}^+ - \frac{23}{4}\sqrt{\frac{2}{35}}f_{20}^+ \right) + 3\sqrt{\frac{2}{35}}f_{03}^+ - 2\sqrt{\frac{7}{15}}f_{11}^+ - \frac{9}{\sqrt{35}}f_{13}^+ + \frac{4}{\sqrt{15}}f_{21}^+ \right] \Big|_{x=a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{21}}f_{00}^- - \frac{2}{\sqrt{7}}f_{02}^- - \frac{13}{\sqrt{245}}f_{04}^- + \frac{3}{\sqrt{14}}f_{10}^- + \frac{34}{7\sqrt{2}}f_{12}^- + \frac{23}{4}\sqrt{\frac{2}{35}}f_{20}^- \right) - 3\sqrt{\frac{2}{35}}f_{03}^- + 2\sqrt{\frac{7}{15}}f_{11}^- + \frac{9}{\sqrt{35}}f_{13}^- - \frac{4}{\sqrt{15}}f_{21}^- \right] \Big|_{x=a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{4\sqrt{35}} = 0 \tag{69}$$

$$\frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{15}}f_{00}^+ - \frac{1}{2\sqrt{5}}f_{02}^+ - \frac{1}{2\sqrt{7}}f_{04}^+ + \frac{5}{2\sqrt{10}}f_{10}^+ - \frac{23\sqrt{2}}{4\sqrt{35}}f_{12}^+ - \frac{15}{\sqrt{2}}f_{20}^+ \right) + \frac{2}{\sqrt{3}}f_{11}^+ - \sqrt{\frac{7}{3}}f_{21}^+ \right] \Big|_{x=a} - \beta \frac{1}{2\alpha} \left[\frac{1}{\sqrt{\pi}} \left(-\frac{1}{2\sqrt{15}}f_{00}^- + \frac{1}{2\sqrt{5}}f_{02}^- + \frac{1}{2\sqrt{7}}f_{04}^- - \frac{5}{2\sqrt{10}}f_{10}^- + \frac{23\sqrt{2}}{4\sqrt{35}}f_{12}^- + \frac{15}{\sqrt{2}}f_{20}^- \right) + \frac{2}{\sqrt{3}}f_{11}^- - \sqrt{\frac{7}{3}}f_{21}^- \right] \Big|_{x=a} - \frac{(1-\beta)\eta}{\alpha\sqrt{\pi}} \frac{1}{\sqrt{15}} = 0 \tag{70}$$

Let introduce the following vectors and matrices:

$$U = (F_{00}, F_{02}, F_{04}, F_{10}, F_{12}, F_{20}, F_{01}, F_{03}, F_{05}, F_{11}, F_{13}, F_{21}),$$

$$J(U, U) = (0, I_{02}, I_{04}, 0, I_{12}, I_{20}, 0, I_{03}, I_{05}, I_{11}, I_{13}, I_{21}),$$

$$A_1 = \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$$

$$A = \frac{1}{\alpha} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{\sqrt{3}} & \frac{3}{\sqrt{5}} & 0 & \frac{2\sqrt{2}}{\sqrt{15}} & 0 & 0 \\ -0 & \frac{4}{\sqrt{7}} & \frac{5}{3} & 0 & -\frac{4\sqrt{2}}{3\sqrt{7}} & 0 \\ -\sqrt{\frac{2}{3}} & 0 & 0 & \sqrt{\frac{5}{3}} & 0 & 0 \\ 0 & -\frac{3\sqrt{2}}{\sqrt{5}\cdot 7} & 0 & \frac{2\sqrt{7}}{\sqrt{3}\cdot 5} & \frac{9}{\sqrt{5}\cdot 7} & -\frac{4}{\sqrt{3}\cdot 5} \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & \sqrt{\frac{7}{3}} \end{bmatrix}$$

$$B = \frac{1}{\alpha\sqrt{\pi}} \begin{bmatrix} \sqrt{2} & \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{105}} - \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} & -\frac{1}{2\sqrt{15}} \\ \sqrt{\frac{2}{3}} & 2\sqrt{2} & \frac{26}{3\sqrt{70}} & -1 & -\frac{2}{\sqrt{7}} & \frac{1}{2\sqrt{5}} \\ -\sqrt{\frac{2}{105}} & \frac{26}{3\sqrt{70}} & -\frac{4}{7\sqrt{2}} & \frac{5}{3\sqrt{35}} & -\frac{13}{\sqrt{245}} & \frac{1}{2\sqrt{7}} \\ -\frac{1}{\sqrt{3}} & -1 & \frac{5}{3\sqrt{35}} & \frac{6}{\sqrt{2}} & \frac{3}{\sqrt{14}} & -\frac{5}{2\sqrt{10}} \\ -\frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{7}} & -\frac{13}{\sqrt{245}} & \frac{3}{\sqrt{14}} & \frac{34}{7\sqrt{2}} & \frac{23\sqrt{2}}{4\sqrt{35}} \\ \frac{1}{2\sqrt{15}} & \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{7}} & -\frac{5}{2\sqrt{10}} & \frac{23\sqrt{2}}{4\sqrt{35}} & \frac{15}{\sqrt{2}} \end{bmatrix}$$

$$G = \left(\frac{1}{2\sqrt{2}}; \frac{1}{4\sqrt{6}}; \frac{1}{4\sqrt{3}}; -\frac{1}{8\sqrt{210}}; \frac{1}{2\sqrt{35}}; \frac{2}{\sqrt{15}} \right)$$

Then the initial boundary value problem for the system of Boltzmann moment Eqs. (11)-(23) with initial condition Eq. (8) and boundary conditions Eqs. (59)-(70) can be written in vector-matrix form:

$$\frac{\partial U}{\partial t} + A_1 \frac{\partial U}{\partial x} = J(U, U), t \in (0, T], x \in (-a, a) \tag{71}$$

$$U|_{t=0} = U_0(x), x \in (-a, a) \tag{72}$$

$$(Ap^+ - Bu^+)|_{x=-a} = \beta(Ap^- + Bu^-)|_{x=-a} + \frac{1}{\alpha\sqrt{\pi}}(1 - \beta)G, t \in (0, T] \tag{73}$$

$$(Ap^+ + Bu^+)|_{x=a} = \beta(Ap^- - Bu^-)|_{x=a} + \frac{1}{\alpha\sqrt{\pi}}(1 - \beta)G, t \in (0, T] \tag{74}$$

where A' - transposed matrix; $U = (u, p)'$, $u = (F_{00}, F_{02}, F_{04}, F_{10}, F_{12}, F_{20})$, $p = (F_{01}, F_{03}, F_{05}, F_{11}, F_{13}, F_{21})$, $u(p)$ - the vector of even (odd) moments on l (second index) of the particle distribution function, $U_0(x) = (F_{00}^0(x), F_{02}^0(x), F_{10}^0(x), F_{04}^0(x), F_{12}^0(x), F_{20}^0(x), F_{01}^0(x), F_{03}^0(x), F_{05}^0(x), F_{11}^0(x), F_{13}^0(x), F_{21}^0(x))'$, - given initial vector functions, $p^+, u^+(p^-, u^-)$ - moments of falling to the boundary (reflected from boundary) particle distribution function.

The matrices A and B are non-singular. So instead of the problem Eqs. (1)-(3), we obtain new problem Eqs. (71)-(74) related to moments of the particle distribution function. The system of equations represents non-linear symmetric system of hyperbolic equations. The initial boundary value problem Eqs. (71)-(74) is studied for the first time.

3. Initial boundary value problem for one-dimensional system of Boltzmann moment equations with boundary conditions of Maxwell-Auzhan

This section discusses the existence and uniqueness of solution to the initial boundary value problem for one-dimensional system of Boltzmann moment equations with boundary conditions of Maxwell-Auzhan in space of functions continuous in time and square-summable over the spatial variable on the method of work.^[37] It is required to find a solution to the system of Eq. (71) satisfying the initial condition (Eqs. (71)) and boundary conditions Eqs. (72) and (73). First, the existence and uniqueness of the solution to

problem Eqs. (71)-(74) is substantiated.

For the problem Eqs. (59)-(73), the following theorem takes place.

Theorem. If $U_0 = (u_0(x), p_0(x)) \in L^2[-a, a]$, then problem Eqs. (71)-(74) has unique solution in $[-a, a] \times [0, T]$, belonging to the space $C([0, T]; L^2[-a, a])$, moreover

$$\|U\|_{C([0, T]; L^2[-a, a])} \leq C_1 \left(\|U_0\|_{L^2[-a, a]} - r_1(0) \right) \tag{75}$$

where C_1 - constant independent from U and $T \sim 0(\|U_0\| - r_1(0))^{-1}$, $r_1(t)$ - particular solution to the Riccati equation (Eq. (79)), that was built in explicit form.

Proof. Let $U_0 \in L^2[-a, a]$. Let's prove estimation Eq. (75). We multiply system of equation (Eq. (71)) by $U = (u, p)'$ and integrate from $-a$ to a :

$$\frac{1}{2} \frac{d}{dt} \int_{-a}^a [(u, u) + (p, p)] dx + \int_{-a}^a \left[\left(A \frac{\partial w}{\partial x}, u \right) + \left(A' \frac{\partial u}{\partial x}, p \right) \right] dx = \int_{-a}^a [(J_1, u) + (J_2, p)] dx \tag{76}$$

After integration by parts, we receive

$$\frac{1}{2} \frac{d}{dt} \int_{-a}^a [(u, u) + (p, p)] dx + (u^-, Ap^-)|_{x=a} - (u^-, Ap^-)|_{x=-a} = \int_{-a}^a [(J_1, u) + (J_2, p)] dx. \tag{77}$$

Taking into account boundary conditions Eqs. (73) and (74), the equality (Eq. (77)) can be written in the following form:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{-a}^a [(u, u) + (p, p)] dx + (Bu^-, u^-)|_{x=a} + \\ & (Bu^-, u^-)|_{x=-a} - \frac{1}{\beta} ((Ap^+ - Bu^+), u^-)|_{x=-a} + \\ & \frac{1}{\beta} ((Ap^+ + Bu^+), u^-)|_{x=a} + (F_1, u^-)|_{x=a} + (F_1, u^-)|_{x=-a} = \\ & \int_{-a}^a [(J_1(u, p), u) + (J_2(u, p), p)] dx \tag{78} \end{aligned}$$

where $F_1 = \frac{(1-\beta)G}{\alpha\beta\sqrt{\pi}}$.

Let's use spherical representation^[38] of a vector $U(t, x) = r(t)\omega(t, x)$, where $\omega(t, x) = (\omega_1(t, x), \omega_2(t, x))$, $r(t) = \|U(t, \cdot)\|_{L^2[-a, a]}$, $\|\omega\|_{L^2[-a, a]} = 1$. Substituting the values $u = r(t)\omega_1(t, x)$, $p = r(t)\omega_2(t, x)$ into Eq. (78), we have that Eq. (79):

$$\frac{dr}{dt} + rP(t) = r^2Q(t) - f(t) \tag{79}$$

where

$$\begin{aligned} P(t) &= (B\omega_1^-, \omega_1^-)|_{x=a} + (B\omega_1^-, \omega_1^-)|_{x=-a} + \\ & + \frac{1}{\beta} [(A\omega_2^+, \omega_1^-)|_{x=a} + (B\omega_1^+, \omega_1^-)|_{x=a} + (B\omega_1^+, \omega_1^-)|_{x=-a} - \\ & (A\omega_2^+, \omega_1^-)|_{x=-a}], \\ Q(t) &= \int_{-a}^a [(J_1(\omega_1, \omega_2), \omega_1) + (J_2(\omega_1, \omega_2), \omega_2)] dx, \\ f(t) &= (F_1, \omega_1^-)|_{x=a} + (F_1, \omega_1^-)|_{x=-a}. \end{aligned}$$

Let's study Eq. (79) with initial condition

$$r(0) = \|U_0\| = \|U_0\|_{L^2[-a, a]} \tag{80}$$

Let $r_1(t)$ is the particular solution to Riccati Eq. (79). Then general solution to Eq. (79) have the form

$$r(t) = r_1(t) + \exp\left(\int_0^t (2Q(\tau)r_1(\tau) - P(\tau)) d\tau\right) \left[C - \right.$$

$$\int_0^t Q(\tau) \exp\left(\int_0^\tau (2Q(\xi)r_1(\xi) - P(\xi))d\xi\right) d\tau \Big]^{-1} \quad (81)$$

Hence taking into consideration initial condition (Eq. (80)), since $z'(0) = 0$, we obtain

$$r(t) = r_1(t) + \exp\left(\int_0^t (2Q(\tau)r_1(\tau) - P(\tau))d\tau\right) \cdot \left[\frac{1}{\|U_0\| - r_1(0)} - \int_0^t Q(\tau) \exp\left(\int_0^\tau (2Q(\xi)r_1(\xi) - P(\xi))d\xi\right) d\tau\right]^{-1} \quad (82)$$

If $R(t) \equiv \int_0^t Q(\tau) \exp\left(\int_0^\tau (2Q(\xi)r_1(\xi) - P(\xi))d\xi\right) d\tau \leq 0, \forall t$, then second term of $r(t)$ is bounded for $\forall t \in [0, +\infty)$. Let $R(t) > 0$. Let denote by T_1 the moment of time at which

$$\frac{1}{\|U_0\| - r_1(0)} - \int_0^{T_1} Q(\tau) \exp\left(\int_0^\tau (2Q(\xi)r_1(\xi) - P(\xi))d\xi\right) d\tau = 0 \quad (83)$$

Then $r(t) - r_1(t)$ is bounded for $\forall t \in [0, T)$, where $T < T_1$ and $T_1 \sim O\left(\frac{1}{\|U_0\| - r_1(0)}\right)$.

Now let's found one particular solution to the equation (Eq. (79)). The non-linear Riccati equation can always be reduced to a second order linear ordinary differential equation^[39]

$$y'' + f_1(t)y' + f_2(t)y = 0 \quad (84)$$

where

$$y(t) = \exp\left(-\int_0^t Q(\tau)r(\tau)d\tau\right), f_1(t) = \frac{Q(t)P(t) - Q'(t)}{Q(t)}, f_2(t) = Q(t)f(t).$$

A solution to this equation will lead to solution to $r(t) = -\frac{y'}{yQ}$ of the original Riccati equation. By substitution

$$y(t) = z(t) \exp\left(-\frac{1}{2}\int_0^t f_1(\tau)d\tau\right) \quad (85)$$

Eq. (31) can be written in following form:^[39]

$$z''(t) + a(t)z(t) = 0 \quad (86)$$

where $a(t) = f_2(t) - \frac{1}{2}f_1'(t) - \frac{1}{4}f_1^2(t)$.

Substituting instead of $f_1(t)$ and $f_2(t)$ values from Eq. (85), we get

$$a(t) = Qf(t) - \frac{2(Q^2P' - Q''Q + Q'^2) + (QP - Q')^2}{4Q^2} \quad (87)$$

In work,^[40] it was proven that function

$$z(t) = 1 + \sum_{k=1}^{\infty} b_k(t) \quad (88)$$

is the solution to Eq. (86), where

$$b_1(t) = -\int_0^t \int_0^\xi a(\tau) d\tau d\xi$$

$$b_k(t) = -\int_0^t \int_0^\xi a(\tau) b_{k-1}(\tau) d\tau d\xi \quad (89)$$

Then

$$y = (1 + \sum_{k=1}^{\infty} b_k(t)) \exp\left(-\frac{1}{2}\int_0^t f_1(\tau)d\tau\right) \quad (90)$$

and

$$r_1(t) = -\frac{y'}{yQ} = -\frac{z'(t) - \frac{1}{2}z(t)f_1(t)}{z(t)Q(t)} \quad (91)$$

where $r_1(t)$ is a particular solution to Riccati equation. The value of

$$r_1(0) = -\frac{z'(0) - \frac{1}{2}z(0)f_1(0)}{z(0)Q(0)} = \frac{f_1(0)}{2Q(0)} \quad (92)$$

Now instead of $f_1(0)$, substitute the value $P(0) - \frac{Q'(0)}{Q(0)}$,

$$\text{we get } r_1(0) = \frac{P(0)Q(0) - Q'(0)}{2Q^2(0)}.$$

The value

$$Q(0) = \int_{-a}^a (J_1(w_1, w_2), w_1) + (J_2(w_1, w_2), w_2) \Big|_{t=0} dx \quad (93)$$

where

$$J_1(w_1, w_2, w_1) \Big|_{t=0} = J_{02}(w_1, w_2, w_{02}) \Big|_{t=0} = J_{02}(w_1^0, w_2^0, w_1^0) = \frac{\sigma_2 - \sigma_0}{2} \left(w_{00}^0 w_{00}^0 - \frac{(w_{01}^0)^2}{\sqrt{3}} \right) w_{02}^0$$

$$w_1 = (w_{00}, w_{02}, w_{10})', w_2 = (w_{01}, w_{03}, w_{11})'$$

$$J_2(w_1, w_2, w_1) \Big|_{t=0} = [J_{03}(w_1, w_2), w_{03}] + [J_{11}(w_1, w_2), w_{11}] \Big|_{t=0} = \left(\frac{1}{4}(\sigma_3 + 3\sigma_1 - 4\sigma_0)w_{00}^0 w_{03}^0 + \frac{1}{4\sqrt{5}}(2\sigma_1 + \sigma_0 - 3\sigma_3)w_{01}^0 w_{02}^0 \right) w_{03}^0 + (\sigma_1 - \sigma_0) \left[w_{00}^0 w_{11}^0 + \frac{1}{2}\sqrt{\frac{5}{3}}w_{10}^0 w_{01}^0 - \sqrt{\frac{2}{15}}w_{01}^0 w_{02}^0 \right] w_{11}^0$$

$$P(0) = (Bw_1^{0-}, w_1^{0-}) \Big|_{x=a} + (Bw_1^{0-}, w_1^{0-}) \Big|_{x=-a} + \frac{1}{\beta} [(Aw_2^{0+}, w_1^{0-}) \Big|_{x=a} + (Bw_1^{0+}, w_1^{0-}) \Big|_{x=a} + (Bw_1^{0+}, w_1^{0-}) \Big|_{x=-a} - (Aw_2^{0+}, w_1^{0-}) \Big|_{x=-a}]$$

Let choose the initial functions such that $Q(0) \neq 0$, then

$r_1(0) = \frac{P(0) - \frac{Q'(0)}{Q(0)}}{2Q(0)}$ is finite number. Expansion form of the function is shown as follows:

$$r_1(t) = \frac{1}{Q(t)} \left[\frac{1}{z(t)} \int_0^t a(\tau)z(\tau)d\tau + \frac{Q(t)P(t) - Q'(t)}{2Q(t)} \right] \quad (94)$$

It is not difficult to prove by immediate substitution that a function $r_1(t)$ satisfies Riccati equation (Eq. (29)).

From Eqs. (36) and (37), we obtain

$$r(t) - r_1(t) \leq C(\|U_0\| - r_1(0)) \quad (95)$$

Hence $\|U(t, \cdot)\|_{L^2[-a, a]} - r_1(t) \leq C(\|U_0\| - r_1(0))$ and $\|U\|_{C([0, T]; L^2[-a, a])} - r_{1C[0, T]} \leq \| \|U(t, \cdot)\|_{L^2[-a, a]} - r_1(t) \| \leq C(\|U_0\| - r_1(0))$.

Let take the segment $\left[0; \frac{1}{\|U_0\| - r_1(0)}\right]$ as interval of solution existence to the problem Eqs. (78) and (79) since integrand $Q(\tau) \exp\left(\int_0^\tau (2Q(\xi)r_1(\xi) - P(\xi))d\xi\right)$ is bounded. Hence, $\forall t \in [0, T]$, where $T < T_1$ takes place a priori estimation Eq. (76).

Now let's prove the existence of a solution for Eqs. (59)-(73) with the help of Galerkin method. Let $\{\omega_1(x)\}_{l=1}^{\infty}$ is a basis in space $L^2[-a, a]$, where dimension of vector $\omega_1(x)$ is equal to dimension of vector U. For each m, we define an approximate solution U_m of Eqs. (71)-(74) as follows:

$$U_m = \sum_{j=1}^m c_{jm}(t)v_j(x) \quad (96)$$

$$\int_{-a}^a \left(\left(\frac{\partial U_m}{\partial t} + A_1 \frac{\partial U_m}{\partial x} \right), v_i(x) \right) dx = \int_{-a}^a (J(U_m), v_i(x)) dx, \quad i = \overline{1, m}, t \in (0, T] \tag{97}$$

$$U_m|_{t=0} = U_{0m}(x), x \in R \tag{98}$$

$$(Ap_m^- \mp Bu_m^-)|_{x=\mp a} = \frac{1}{\beta} (Ap_m^+ \pm Bu_m^+)|_{x=\pm a} \pm \frac{(1-\beta)\eta}{\alpha\beta\sqrt{\pi}} G \tag{99}$$

where U_{0m} is the orthogonal projection in L^2 of function U_0 on the subspace, spanned by v_1, \dots, v_m , $J(U_m) = (J_1(u_m, p_m), J_2(u_m, p_m))$.

Represent $v_j(x)$ in the form $v_j(x) = (v_j^{(1)}, v_j^{(2)})'$, where $v_j^{(1)} = (v_{j1}, v_{j2}, v_{j3})'$, $v_j^{(2)} = (v_{j4}, v_{j5}, v_{j6})'$.

The coefficients $c_{jm}(t)$ are determined from the equations

$$\begin{aligned} \sum_{j=1}^m \left\{ \frac{dc_{jm}}{dt} \int_{-a}^a (v_j, v_i) dx + c_{jm} \left[(Bv_j^{-(1)}, v_i^{-(1)}) \Big|_{x=a} + (Bv_j^{-(1)}, v_i^{-(1)}) \Big|_{x=-a} + (Bv_j^{-(1)}, v_j^{-(1)}) \Big|_{x=-a} + \frac{1}{\beta} \left((Av_i^{+(2)} + Bv_i^{+(1)}), v_j^{-(1)} \right) \Big|_{x=a} - \frac{1}{\beta} \left((Av_i^{+(2)} - Bv_i^{+(1)}), v_j^{-(1)} \right) \Big|_{x=-a} + \frac{1}{\beta} \left((Av_j^{+(2)} + Bv_j^{+(1)}), v_i^{-(1)} \right) \Big|_{x=a} - \frac{1}{\beta} \left((Av_j^{+(2)} - Bv_j^{+(1)}), v_i^{-(1)} \right) \Big|_{x=-a} + (F_1, v_i^{-(1)}) \Big|_{x=a} + (F_1, v_i^{-(1)}) \Big|_{x=-a} + (F_1, v_j^{-(1)}) \Big|_{x=a} + (F_1, v_j^{-(1)}) \Big|_{x=-a} - \int_{-a}^a \left(A \frac{\partial v_i^{(2)}}{\partial x}, v_j^{(1)} \right) + \left(A \frac{\partial v_i^{(2)}}{\partial x}, v_j^{(2)} \right) dx \right\} = \int_{-a}^a (J(\sum_{j=1}^m c_{jm} v_j), v_i) dx, i = \overline{1, m}, t \in (0, T], c_{im}(0) = d_{im}, i = \overline{1, m} \tag{100} \end{aligned}$$

where $i = \overline{1, m}, t \in (0, T], c_{im}(0) = d_{im}, i = \overline{1, m}, d_{im}$ is the i -th component of U_{0m} .

Multiply Eq. (97) by $c_{im}(t)$ and sum over i from 1 to m :

$$\int_{-a}^a \left(\left(\frac{\partial U_m}{\partial t} + A_1 \frac{\partial U_m}{\partial x} \right), U_m \right) dx = \int_{-a}^a (J(U_m), U_m) dx \tag{101}$$

With the help of above shown arguments, let's prove that $r_m(t)$ is bounded in some time interval $[0, T_m]$, where $U_m(t, x) = r_m(t)\omega_m(t, x), T_m \approx 0 \left((\|U_{0m}\| - r_{m1}(0))^{-1} \right), T_m \geq T \forall m$, and $\|U_m\|_{C([0, T]; L^2[-a, a])} \leq C_2 (\|U_0\|_{L^2[-a, a]} - r_1(0))$, where C_2 - constant and independent from m .

Then, the solution to the system of Eqs. (96)-(99) or Eq. (100) follows from estimation Eq. (41). Thus, the sequence $\{U_m\}$ of approximate solutions of problem Eqs. (5)-(7) is uniformly bounded in function space $([0, T]; L^2[-a, a])$. Moreover, homogeneous system of equations $\tau E + \frac{1}{\alpha} A \xi$ with respect to τ, ξ has only trivial solution. Then, it follows from results in Ref. [41] that $U_m \rightarrow U$ is weak in $C([0, T]; L^2[-a, a])$ and $J(U_m) \rightarrow J(U)$ is weak in $C([0, T]; L^2[-a, a])$ as $m \rightarrow \infty$. Further, it can be shown by standard methods that limit element is a weak solution to the problem Eqs. (72)-(74). The theorem is proven.

3. Numerical solution to the initial boundary value problem for one-dimensional system of Boltzmann moment equations with boundary conditions of Maxwell-Auzhan

It is impossible to find an exact solution to the problem Eqs. (71)-(74), so we use numerical method to construct an approximate solution to the problem Eqs. (71)-(74). The differential problem is approximated by a finite-difference scheme, and an algorithm and a program for numerical implementation on a computer are compiled. At the end, the graphs of the moments of the particle distribution function are given.

Let us construct a finite-difference scheme for the numerical solution to problems Eqs. (71)-(74). Let divide the segment $[0, T]$ into N_1 equal parts and the segment $[-a, a]$ into N_2 equal parts. Let introduce the grid functions $u_{ij} = u(t_i, x_j), p_{ij} = p(t_i, x_j)$. Let us approximate the differential problem Eqs. (71)-(74) by the following finite difference scheme:

$$\frac{u_{i+1,j}^{n+1} - u_{ij}^{n+1}}{\tau} + A \frac{p_{ij}^{n+1} - p_{i,j-1}^{n+1}}{h} = J_1(u_{ij}^n, p_{ij}^n) \quad i = 0, 1, \dots, N_1 - 1; j = 1, \dots, N_2 \tag{102}$$

$$\frac{p_{i+1,j}^{n+1} - p_{ij}^{n+1}}{\tau} + A \frac{u_{i,j+1}^{n+1} - u_{ij}^{n+1}}{h} = J_2(u_{ij}^n, p_{ij}^n), \quad i = 0, 1, \dots, N_1 - 1; j = N_2 - 1, \dots, 0; \tag{103}$$

$$u_{0j}^{n+1} = u_j^0, p_{0j}^{n+1} = w_j^0, j = 0, 1, \dots, N_2 \tag{104}$$

$$(Ap^- + Bu^-)_{i,0}^{n+1} = \frac{1}{\beta} (Ap^+ - Bu^+)_{i,0}^n - \frac{\pi(1-\beta)}{\alpha^3\beta} G, \quad i = 0, 1, \dots, N_1 \tag{105}$$

$$(Ap^- - Bu^-)_{i,N_2}^{n+1} = \frac{1}{\beta} (Ap^+ + Bu^+)_{i,N_2}^n + \frac{\pi(1-\beta)}{\alpha^3\beta} G, \quad i = 0, 1, \dots, N_1 \tag{106}$$

τ - time step, h - space variable step.

In order to find a numerical solution to the problem Eqs. (102)-(106) and (78)-(84), we use the iterative method.

At $i = 0, j = 0, 1, \dots, N_2$, the solution to the problem can be determined using the initial conditions Eq. (80). Let $i = 0, j = 1$. With the help of Eq. (77), we obtain the following Eq. (107):

$$\frac{u_{11}^{n+1} - u_{01}^{n+1}}{\tau} + A \frac{p_{01}^{n+1} - w_{00}^{n+1}}{h} = J_1(u_{01}^n, p_{01}^n) \tag{107}$$

where $u_{11}^{n+1} = u_{01}^{n+1} - \frac{\tau}{h} A(p_{01}^{n+1} - p_{00}^{n+1}) + \tau J_1(u_{01}^n, p_{01}^n)$.

Let $i = 0, j = 2$. In this case, we get the following ratio in Eq. (108):

$$u_{12}^{n+1} = u_{02}^{n+1} - \frac{\tau}{h} A(p_{02}^{n+1} - p_{01}^{n+1}) + \tau J_1(u_{02}^n, p_{02}^n) \tag{108}$$

And so on, at $i = 0, j = N_2$ we get the following ratio in Eq. (109):

$$u_{1N_2}^{n+1} = u_{0N_2}^{n+1} - \frac{\tau}{h} A(p_{0N_2}^{n+1} - p_{0,N_2-1}^{n+1}) + \tau J_1(u_{0N_2}^n, p_{0N_2}^n) \tag{109}$$

Using the value of u_{1,N_2}^{n+1} , from the boundary condition Eq. (84), we find ($i = 1, j = N_2$) in Eq. (110):

$$p_{1,N_2}^{n+1} = A^{-1} \left[\frac{1}{\beta} (Ap_{1,N_2}^n + Bu_{1,N_2}^n) + \frac{\pi(1-\beta)}{\alpha^3\beta} G + Bu_{1,N_2}^{n+1} \right] \quad (110)$$

In this way, we found all the values of the vector u_{1j}^{n+1} on the first layer of the difference grid, except for u_{10}^{n+1} .

To determine the values of p_{1j}^{n+1} we use Eq. (79) and perform the calculations from right to left. If $i = 0, j = N_2 - 1$ we get Eqs. (111)-(113)

$$p_{1,N_2-1}^{n+1} = p_{0,N_2-1}^{n+1} - \frac{\tau}{h} A'(u_{0,N_2}^{n+1} - u_{0,N_2-1}^{n+1}) + \tau J_2(u_{0,N_2-1}^n, p_{0,N_2-1}^n) \quad (111)$$

At $i = 0, j = N_2 - 2$ we have

$$p_{1,N_2-2}^{n+1} = p_{0,N_2-2}^{n+1} - \frac{\tau}{h} A'(u_{0,N_2-1}^{n+1} - u_{0,N_2-2}^{n+1}) + \tau J_2(u_{0,N_2-2}^n, p_{0,N_2-2}^n) \quad (112)$$

And so on. At $i = 0, j = 0$, we get the following correlation

$$p_{1,0}^{n+1} = p_{0,0}^{n+1} - \frac{\tau}{h} A'(u_{0,1}^{n+1} - u_{0,0}^{n+1}) + \tau J_2(u_{0,0}^n, p_{0,0}^n) \quad (113)$$

Using the found value from $p_{1,0}^{n+1}$ the boundary condition Eq. (80), we find ($i = 1, j = 0$) in Eq. (114):

$$u_{1,0}^{n+1} = B^{-1} \left[\frac{1}{\beta} (Ap_{1,0}^n - Bu_{1,0}^n) - \frac{\pi(1-\beta)}{\alpha^3\beta} G - Ap_{1,0}^{n+1} \right] \quad (114)$$

Thus, we have defined all the values $u_{1,j}^{n+1}$ and $p_{1,j}^{n+1}$ on the first layer of the difference grid, including the left and right boundaries of the interval $(-a, a)$. Let's continue this process for the values of $i = 1, \dots, N_1 - 1$ and define the values $u_{i,j}^{n+1}$ and $p_{i,j}^{n+1}$ at all nodes of the difference grid.^[42-47]

Let start the iterative process for n and continue the calculations until we achieve the following conditions: $|u_{ij}^{n+1} - u_{ij}^n| < \varepsilon, |p_{ij}^{n+1} - p_{ij}^n| < \varepsilon, i = 0, 1, \dots, N_1 - 1; j = 1, \dots, N_2$, where ε - a given sufficiently small number.

The numerical experiment was carried out with the

following data: $[-a, a] \cong [0,1], \sigma_0 = 120, \sigma_2 = -24$.

Segment $[0,1]$ is divided into 10 equal parts, step $h = 0.1$ in space variable x . Let denote by τ the time step and choose $\tau = 0.02$. The inequality $\frac{\tau}{h} < 1$ ($\tau=0.02, h=0.1$) satisfies the stability condition of the finite-difference scheme. Fulfilment of the stability condition and analysis of the results of the numerical experiment confirms the convergence of the difference scheme.

The initial functions are chosen as follows: $F_{00} = 1 - x; F_{02} = 1 - x/2; F_{04} = x(1 - x); F_{10} = x - 1/2; F_{12} = (1 - x)/2; F_{20} = x(1 - x)/2; F_{01} = x; F_{02} = x - 2; F_{05} = x(1 - x); F_{11} = x + 1; F_{12} = 1 - x/3; F_{21} = x(2 - x)/2; \beta = 1; \alpha = 5.41334827830624; \Theta = 346 \text{ K}$, Boltzmann constant = $8.617333 \cdot 10^{-5} \text{ eV/K}$, $\beta = 1; \sigma_0 = 120; \sigma_1 = 0; \sigma_2 = -24; \sigma_3 = 0; \sigma_4 = 0; \sigma_5 = 0; u = (F_{00}, F_{02}, F_{04}, F_{10}, F_{12}, F_{20})$, $p = (F_{01}, F_{03}, F_{05}, F_{11}, F_{13}, F_{21})$, where $[0; 1]$ - x axis segment, $[0; 0,2]$ - t axis segment, the vertical axis shows the value of distribution function moments. $F_{00} = \rho$ - gas density; $F_{01} = \alpha\rho V$ - the production of gas density and the average velocity of gas and $\alpha = 1/\sqrt{R\Theta}$, where R - the Boltzmann constant, Θ - the temperature of wall; $F_{02}, F_{03}, \dots, F_{21}$ will be expressed through the hydrodynamic characteristics of the gas.

We will further present tables of values and graphs of the particle distribution moments: $F_{00} = \rho$ - gas density; $F_{01} = \alpha\rho V$ - the production of gas density and the average velocity of gas and $\alpha = 1/\sqrt{R\Theta}$, where R - the Boltzmann constant, Θ - the temperature of wall; $F_{02}, F_{03}, \dots, F_{21}$ will be expressed through the hydrodynamic characteristics of the gas. We will further present tables of values and Figs. 2-13 of the particle distribution moments. $F_{02} = \frac{\alpha^2}{2\sqrt{3}}(P_{33} + \rho V^2)$, $F_{03} = \frac{\alpha^3}{\sqrt{15}}(2q + 9V\rho R\theta + V^3\rho)$, $F_{10} = \alpha\sqrt{\frac{5}{2}}\rho V - \sqrt{\frac{2}{5}}\alpha^3\left(q + VP_{33} + \rho V\left(\frac{3P}{2\rho} + \frac{V^2}{2}\right)\right)$, and etc. However, they do not possess a definite physical meaning.

F_{00}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	1	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0
t_2 0.02	1.0000	0.8947	0.7947	0.6947	0.5947	0.4947	0.3947	0.2947	0.1947	0.0947	-0.0053
t_3 0.04	1.0000	0.9892	0.7895	0.6895	0.5895	0.4895	0.3895	0.2895	0.1895	0.0895	-0.0109
t_4 0.06	1.0000	0.9885	0.7792	0.6842	0.5842	0.4842	0.3842	0.2842	0.1842	0.0842	-0.0170
t_5 0.08	1.0000	1.0877	0.7684	0.6792	0.5790	0.4790	0.3790	0.2790	0.1790	0.0790	-0.0237
t_6 0.1	1.0000	1.0920	0.7522	0.6745	0.5737	0.4737	0.3737	0.2737	0.1737	0.0737	-0.0311
t_7 0.12	1.0000	1.1964	0.7348	0.6703	0.5684	0.4684	0.3684	0.2684	0.1684	0.0684	-0.0394
t_8 0.14	1.0000	1.2061	0.7114	0.6669	0.5630	0.4632	0.3632	0.2632	0.1632	0.0632	-0.0487
t_9 0.16	1.0000	1.3164	0.6863	0.6644	0.5576	0.4579	0.3579	0.2579	0.1579	0.0579	-0.0591
t_{10} 0.18	1.0000	1.3321	0.6544	0.6630	0.5520	0.4527	0.3527	0.2527	0.1527	0.0527	-0.0710

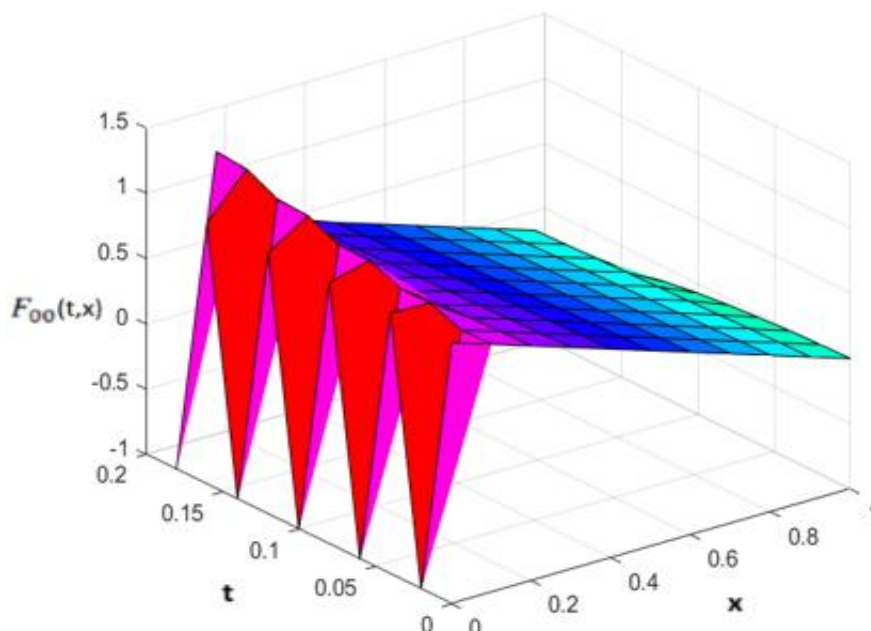


Fig. 2: F_{00} - gas density.

F_{01}

$\begin{matrix} x \\ t \end{matrix}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
t_1	0	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1
t_2	0.02	0	0.0994	0.1994	0.2994	0.3994	0.4994	0.5994	0.6994	0.7994	0.8994	1.1349
t_3	0.04	0	0.0040	0.1987	0.2987	0.3987	0.4987	0.5987	0.6987	0.7987	0.8987	1.3234
t_4	0.06	0	0.0040	0.1970	0.2979	0.3979	0.4979	0.5979	0.6979	0.7979	0.8979	1.5775
t_5	0.08	0	-0.0910	0.1955	0.2966	0.3969	0.4969	0.5969	0.6969	0.7969	0.8969	1.9058
t_6	0.1	0	-0.0908	0.1936	0.2948	0.3958	0.4958	0.5958	0.6958	0.7958	0.8957	2.3126
t_7	0.12	0	-0.1858	0.1924	0.2921	0.3946	0.4946	0.5946	0.6945	0.7945	0.8945	2.7974
t_8	0.14	0	-0.1857	0.1911	0.2886	0.3933	0.4933	0.5932	0.6931	0.7931	0.8930	3.3553
t_9	0.16	0	-0.2811	0.1910	0.2840	0.3918	0.4918	0.5917	0.6916	0.7915	0.8914	3.9788
t_{10}	0.18	0	-0.2816	1.1914	0.2781	0.3903	0.4902	0.5917	0.6900	0.7898	0.8897	4.6592

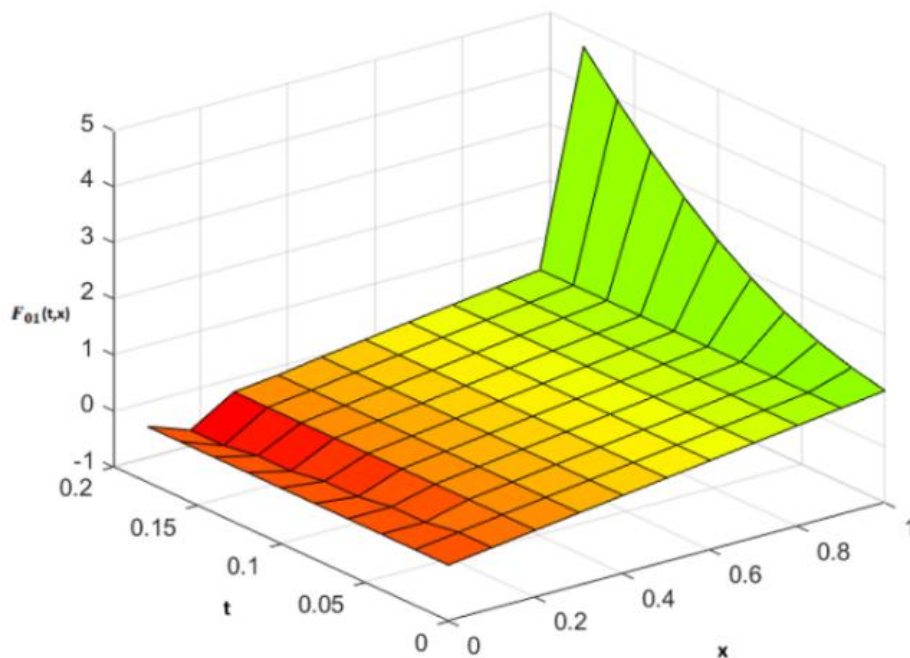


Fig. 3: F_{01} - the production of gas density, the average velocity of gas and $\alpha = 1/\sqrt{R\theta}$.

$$F_{02}$$

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
t_1	0	1	0.9500	0.9000	0.8500	0.8000	0.7500	0.7000	0.6500	0.6000	0.5500	0.5000
t_2	0.02	1.0000	0.9141	0.8700	0.8260	0.7821	0.7382	0.6944	0.6506	0.6069	0.5632	0.5196
t_3	0.04	1.0000	0.9910	0.8414	0.8031	0.7650	0.7271	0.6893	0.6516	0.6142	0.5768	0.5270
t_4	0.06	1.0000	0.9688	0.8053	0.7813	0.7488	0.7166	0.6847	0.6531	0.6218	0.5908	0.5224
t_5	0.08	1.0000	1.0457	0.7702	0.7610	0.7334	0.7067	0.6806	0.6549	0.6297	0.6051	0.5088
t_6	0.1	1.0000	1.0233	0.7275	0.7422	0.7187	0.6975	0.6770	0.6572	0.6381	0.6197	0.4918
t_7	0.12	1.0000	1.0999	0.6856	0.7255	0.7046	0.6888	0.6738	0.6598	0.6468	0.6348	0.4815
t_8	0.14	1.0000	1.0769	0.6359	0.7108	0.6911	0.6807	0.6711	0.6628	0.6558	0.6502	0.4924
t_9	0.16	1.0000	1.1527	0.5868	0.6988	0.6780	0.6731	0.6688	0.6662	0.6652	0.6660	0.5434
t_{10}	0.18	1.0000	1.1284	0.5296	0.6895	0.6652	0.6661	0.6670	0.6700	0.6750	0.6822	0.6573

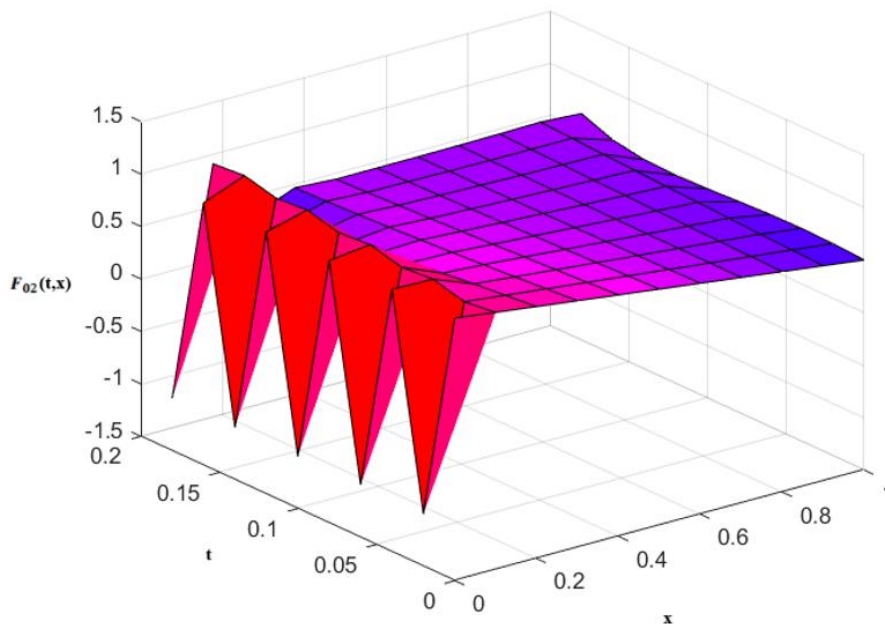


Fig. 4: F_{02} - the particle distribution moments.

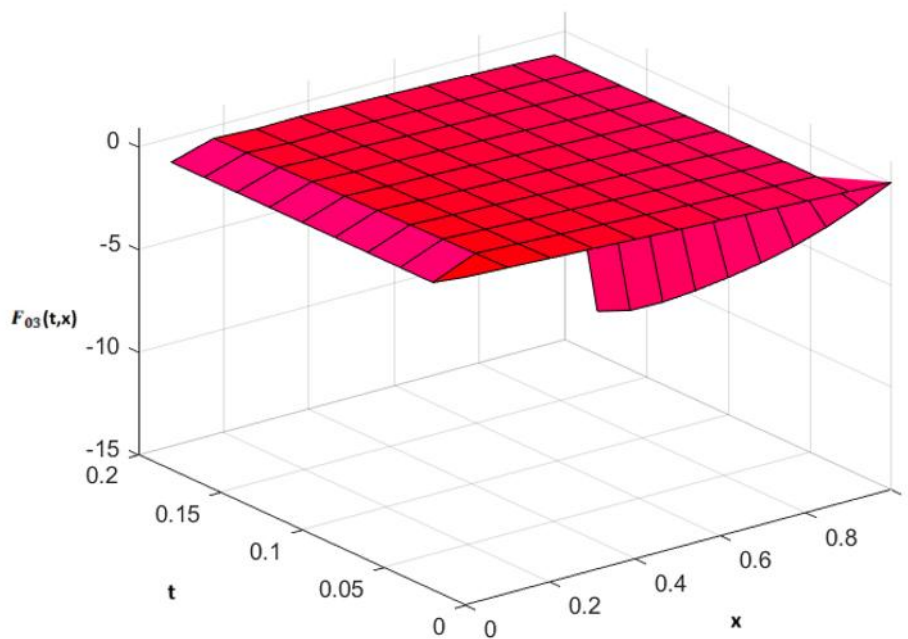


Fig. 5: F_{03} - the particle distribution moments.

F_{03}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	0.5000	0.5500	0.6000	0.6500	0.7000	0.7500	0.8000	0.8500	0.9000	0.9500	0
t_2 0.02	0	0.5467	0.5967	0.6468	0.6969	0.7471	0.7972	0.8473	0.8974	0.9476	-0.8785
t_3 0.04	0	0.5207	0.5935	0.6437	0.6939	0.7441	0.7944	0.8446	0.8949	0.9452	-2.5826
t_4 0.06	0	0.4943	0.5890	0.6405	0.6908	0.7412	0.7915	0.8419	0.8923	0.9427	-4.0837
t_5 0.08	0	0.4710	0.5834	0.6373	0.6878	0.7382	0.7887	0.8392	0.8898	0.9403	-5.3541
t_6 0.1	0	0.4471	0.5769	0.6340	0.6848	0.7353	0.7859	0.8365	0.8872	0.9379	-6.3682
t_7 0.12	0	0.4262	0.5694	0.6306	0.6817	0.7324	0.7831	0.8338	0.8846	0.9355	-7.1018
t_8 0.14	0	0.4046	0.5612	0.6270	0.6786	0.7295	0.7803	0.8312	0.8821	0.9331	-7.5331
t_9 0.16	0	0.3859	0.5524	0.6231	0.6755	0.7265	0.7775	0.8285	0.8795	0.9306	-7.6427
t_{10} 0.18	0	0.3664	0.5429	0.6191	0.6724	0.7236	0.7747	0.8258	0.8770	0.9282	-7.4133

F_{04}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0
t_2 0.02	2.7756e-16	0.0937	0.1628	0.2120	0.2411	0.2501	0.2392	0.2083	0.1574	0.0864	-0.0045
t_3 0.04	-6.0162e-16	0.1073	0.1656	0.2139	0.2421	0.2502	0.2384	0.2065	0.1574	0.0828	-0.0108
t_4 0.06	3.8858e-16	0.1212	0.1684	0.2157	0.2430	0.2503	0.2375	0.2047	0.1519	0.0791	-0.0189
t_5 0.08	-2.7756e-16	0.1354	0.1709	0.2176	0.2439	0.2502	0.2365	0.2028	0.1491	0.0754	-0.0288
t_6 0.1	5.5511e-16	0.1499	0.1733	0.2294	0.2448	0.2502	0.2355	0.2009	0.1463	0.0716	-0.0404
t_7 0.12	-5.5511e-16	0.1648	0.1754	0.2213	0.2456	0.2500	0.2345	0.19890	0.1434	0.0678	-0.0538
t_8 0.14	4.4409e-16	0.1801	0.1772	0.2232	0.2463	0.2499	0.2334	0.1969	0.1404	0.0639	-0.0688
t_9 0.16	-2.7756e-16	0.1958	0.1785	0.2252	0.2470	0.2497	0.2322	0.1948	0.1374	0.0600	-0.0855
t_{10} 0.18	4.4409e-16	0.2120	0.1793	0.2274	0.2476	0.2494	0.2310	0.1927	0.1343	0.0560	-0.1037

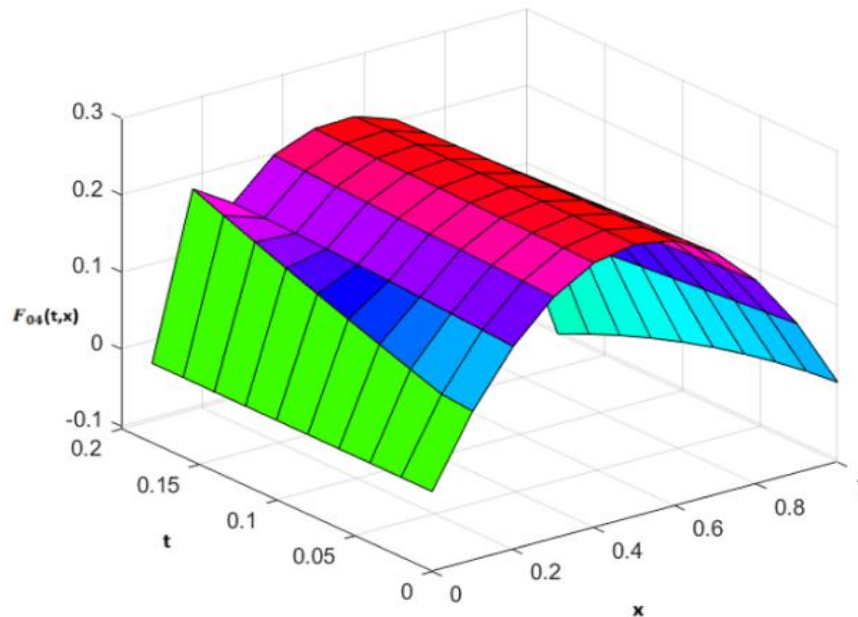


Fig. 6: F_{04} - the particle distribution moments.

F_{05}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0
t_2 0.02	0	0.0740	0.1449	0.1964	0.2283	0.2407	0.2334	0.2064	0.1596	0.0930	1.4706
t_3 0.04	0	0.0199	0.1307	0.1834	0.2170	0.2315	0.2268	0.2026	0.1589	0.0957	2.8687
t_4 0.06	0	0.0165	0.1174	0.1709	0.2061	0.2225	0.2201	0.1987	0.1580	0.0980	4.1189
t_5 0.08	0	-0.0353	0.1050	0.1589	0.1954	0.2137	0.2135	0.1946	0.1569	0.1001	5.1557
t_6 0.1	0	-0.0366	0.0940	0.1473	0.1851	0.2050	0.2069	0.1905	0.1555	0.1018	5.9316
t_7 0.12	0	-0.0865	0.0839	0.1361	0.1751	0.1965	0.2003	0.1862	0.1540	0.1032	6.4269
t_8 0.14	0	-0.0860	0.0754	0.1253	1.1654	0.1881	0.1937	0.1818	0.1522	0.1043	6.6582
t_9 0.16	0	-0.1342	0.0679	0.1148	0.1559	0.1798	0.1871	0.1774	0.1502	0.1052	6.6884
t_{10} 0.18	0	-0.1321	0.0623	0.1045	0.1467	0.1717	0.1805	0.1728	0.1480	0.1057	6.6335

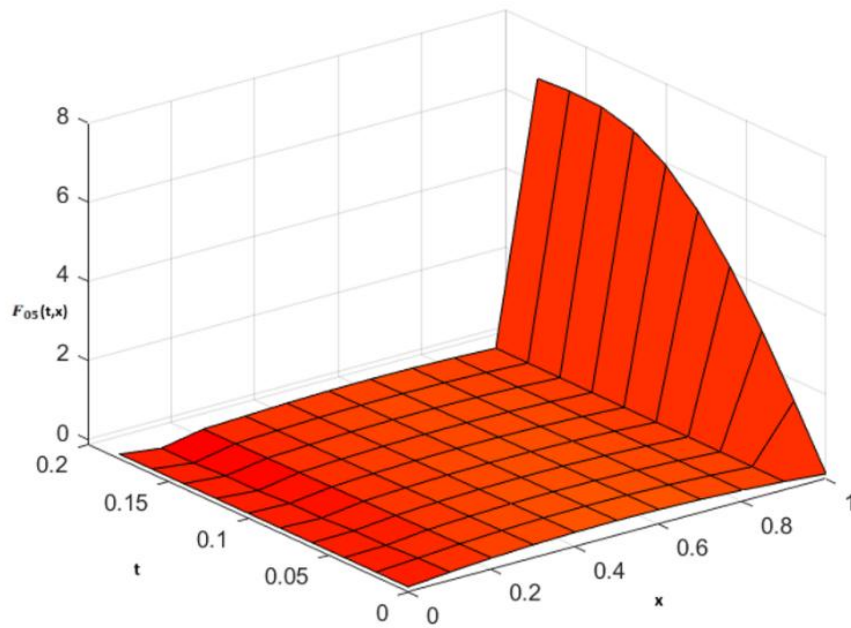


Fig. 7: F_{05} - the particle distribution moments.

$$F_{10}$$

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
t_1	0	0.5000	-0.4000	-0.3000	-0.2000	-0.1000	0	0.1000	0.2000	0.3000	0.4000	0.5000
t_2	0.02	0.5000	-0.4709	-0.3709	-0.2709	-0.1709	-0.0709	0.0291	0.1291	0.2291	0.3291	0.4291
t_3	0.04	0.5000	-0.5180	-0.4430	-0.3432	-0.2433	-0.1435	-0.0436	0.0562	0.1561	0.2559	0.3215
t_4	0.06	0.5000	-0.5684	-0.5156	-0.4170	-0.3174	-0.2179	-0.1184	-0.0189	0.0806	0.1800	0.1817
t_5	0.08	0.5000	-0.6201	-0.5895	-0.4924	-0.3935	-0.2944	-0.1954	-0.0964	0.0025	-0.1013	-0.0172
t_6	0.1	0.5000	-0.6753	-0.6641	-0.5695	-0.4716	-0.3732	-0.2749	-0.1766	-0.0785	-0.0196	-0.1618
t_7	0.12	0.5000	-0.7324	-0.7402	-0.6483	-0.5520	-0.4544	-0.3570	-0.2597	-0.1625	-0.0656	-0.3435
t_8	0.14	0.5000	-0.7935	-0.8171	-0.7289	-0.6348	-0.5383	-0.4419	-0.3458	-0.2499	-0.1543	-0.5143
t_9	0.16	0.5000	-0.8569	-0.8958	-0.8114	-0.7203	-0.6250	-0.5299	-0.4353	-0.3409	-0.2469	-0.6604
t_{10}	0.18	0.5000	-0.9249	-0.9755	-0.8960	-0.8086	-0.7147	-0.6213	-0.5283	-0.4358	-0.3437	-0.7676

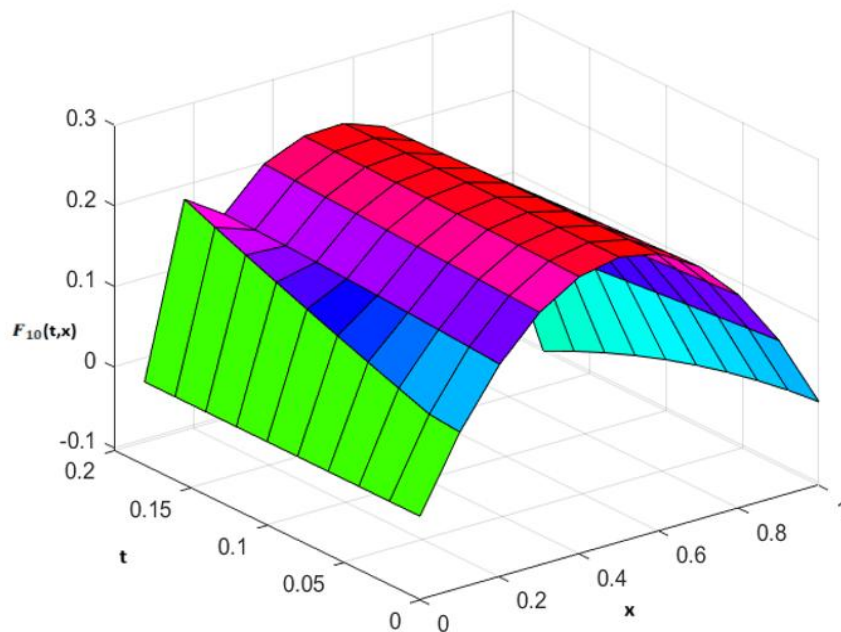


Fig. 8: F_{10} - the particle distribution moments.

F_{11}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
t_1	0	1	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.9000	2
t_2	0.02	0	1.0532	1.1559	1.2587	1.3617	1.4648	1.5680	1.6714	1.7749	1.8785	3.5024
t_3	0.04	0	1.0268	1.1137	1.2193	1.3252	1.4313	1.5378	1.6447	1.7518	1.8593	4.7662
t_4	0.06	0	0.9784	1.0661	1.1817	1.2904	1.3997	1.5095	1.6199	1.7308	1.8422	5.6864
t_5	0.08	0	0.9603	1.0196	1.1460	1.2574	1.3698	1.4830	1.5970	1.7118	1.8274	6.1781
t_6	0.1	0	0.9201	0.9676	1.1121	1.2260	1.3416	1.4582	1.5759	1.6947	1.8147	6.1826
t_7	0.12	0	0.9098	0.9167	1.0800	1.1962	1.3150	1.4251	1.5567	1.6797	1.8043	5.6699
t_8	0.14	0	0.8772	0.8599	1.0498	1.1677	1.2900	1.4137	1.5393	1.6667	1.7961	4.6360
t_9	0.16	0	0.8744	0.8044	1.0216	1.1406	1.2666	1.3940	1.5237	1.6557	1.7961	3.0970
t_{10}	0.18	0	0.8490	0.7423	0.9951	1.1147	1.2446	1.3759	1.5099	1.6468	1.7866	1.0820

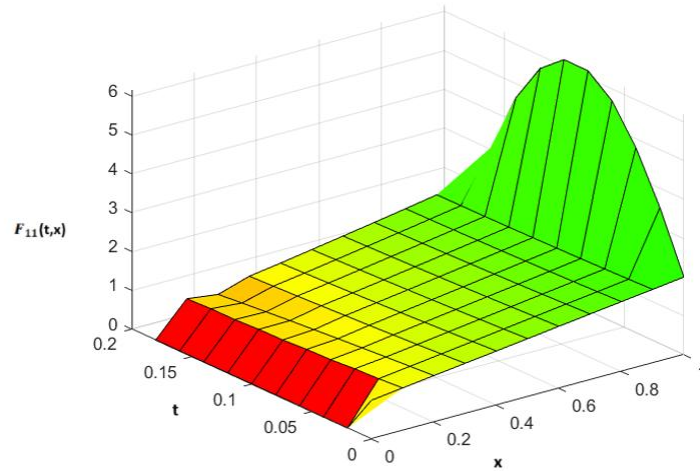


Fig. 9: F_{11} - the particle distribution moments.

F_{12}

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
t_1	0	0.5000	0.4500	0.4000	0.3500	0.3000	0.2500	0.2000	0.1500	0.1000	0.0500	0
t_2	0.02	-0.5000	0.4477	0.3977	0.3477	0.2976	0.2476	0.1976	0.1476	0.0975	0.0475	-0.0025
t_3	0.04	0.5000	0.4841	0.3954	0.3453	0.2953	0.2452	0.1952	0.1451	0.0951	0.0450	-0.0051
t_4	0.06	-0.5000	0.4729	0.3906	0.3430	0.2929	0.2428	0.1928	0.1427	0.0926	0.0425	-0.0076
t_5	0.08	0.5000	0.5117	0.3856	0.3408	0.2905	0.2404	0.1903	0.1402	0.0901	0.0400	-0.0098
t_6	0.1	-0.5000	0.5028	0.3780	0.3386	0.2882	0.2380	0.1879	0.1378	0.0876	0.0375	-0.0114
t_7	0.12	0.5000	0.5441	0.3700	0.3366	0.2858	0.2357	0.1855	0.1353	0.0852	0.0350	-0.0123
t_8	0.14	-0.5000	0.5378	0.3593	0.3349	0.2834	0.2333	0.1831	0.1329	0.0827	0.0325	-0.0122
t_9	0.16	0.5000	0.5816	0.3480	0.3335	0.2809	0.2309	0.1807	0.1304	0.0802	0.0300	-0.0109
t_{10}	0.18	-0.5000	0.5780	0.3336	0.3325	0.2784	0.2285	0.1782	0.1280	0.0778	0.0275	-0.0080

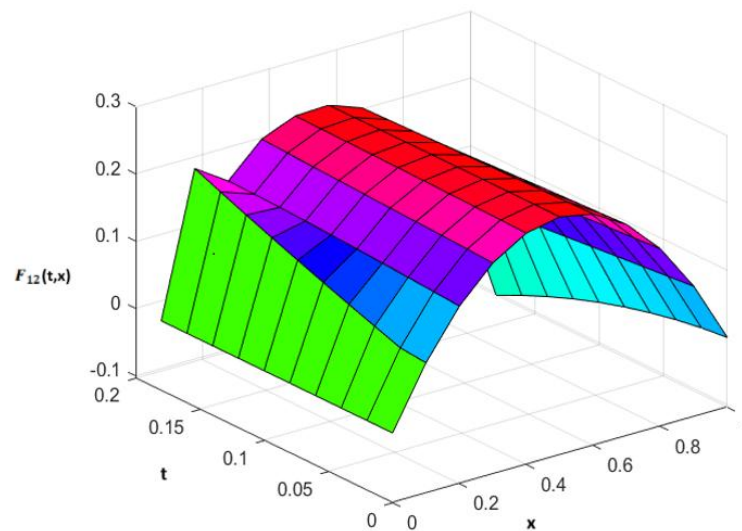


Fig. 10: F_{12} - the particle distribution moments.

$$F_{13}$$

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	1	0.9667	0.9333	0.9000	0.8667	0.8333	0.8000	0.7667	0.7333	0.7000	0.6666
t_2 0.02	0	0.9671	0.9339	0.9007	0.8674	0.8342	0.8009	0.7676	0.7344	0.7011	-0.9223
t_3 0.04	0	0.9152	0.9344	0.9013	0.8682	0.8350	0.8018	0.7686	0.7354	0.7022	-2.3731
t_4 0.06	0	0.8697	0.9324	0.9020	0.8689	0.8358	0.8027	0.7696	0.7365	0.7033	-3.6597
t_5 0.08	0	0.8226	0.9283	0.9025	0.8696	0.8366	0.8036	0.7706	0.7375	0.7045	-4.7575
t_6 0.1	0	0.7817	0.9221	0.9028	0.8704	0.8375	0.8045	0.7715	0.7385	0.7056	-5.6436
t_7 0.12	0	0.7389	0.9142	0.9028	0.8711	0.8383	0.8054	0.7725	0.7396	0.7067	-6.2969
t_8 0.14	0	0.7021	0.9046	0.9023	0.8718	0.8391	0.8063	0.7735	0.7406	0.7078	-6.6987
t_9 0.16	0	0.6632	0.8938	0.9015	0.8724	0.8399	0.8072	0.7744	0.7416	0.7088	-6.8320
t_{10} 0.18	0	0.6301	0.8817	0.9001	0.8729	0.8407	0.8080	0.7754	0.7426	0.7099	-6.6813

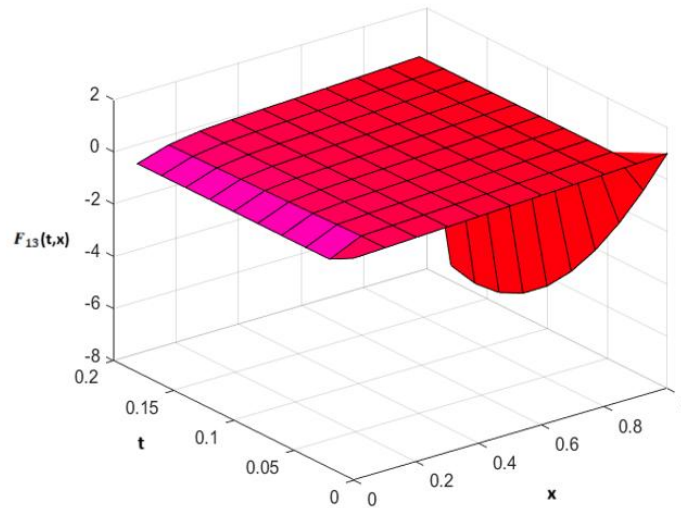


Fig. 11: F_{13} - the particle distribution moments.

$$F_{20}$$

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	0	0.0450	0.0800	0.1050	0.1200	0.1250	0.1200	0.1050	0.0800	0.0450	0
t_2 0.02	2.7575e-17	0.0520	0.0811	0.1015	0.1130	0.1155	0.1089	0.0930	0.677	0.0329	-0.0116
t_3 0.04	0	0.0822	0.0840	0.0994	0.1070	0.1066	0.0978	0.0805	0.0543	0.0190	-0.0429
t_4 0.06	2.7575e-17	0.1147	0.0882	0.0993	0.1026	0.0987	0.0872	0.0678	0.0401	0.0036	-0.0984
t_5 0.08	-5.5511e-17	0.1488	0.0939	0.1017	0.1001	0.0923	0.0775	0.0554	0.0254	-0.0130	-0.1774
t_6 0.1	8.27266e-17	0.1854	0.1015	0.1072	0.1000	0.0878	0.0692	0.0438	0.0108	-0.0302	-0.2714
t_7 0.12	-1.3788e-16	0.2244	0.1113	0.1164	0.1027	0.0857	0.0628	0.0334	-0.0032	-0.0478	-0.3607
t_8 0.14	1.6445e-16	0.2662	0.1236	0.1300	0.1089	0.0866	0.0587	0.0247	-0.0162	-0.0650	-0.4139
t_9 0.16	-1.7924e-16	0.3112	0.1389	0.1488	0.1189	0.0910	0.0577	0.0185	-0.0275	-0.0814	-0.3888
t_{10} 0.18	2.0681-e16	0.3595	0.1576	0.1733	0.1334	0.0998	0.0604	0.0154	-0.0356	-0.0926	-0.2375

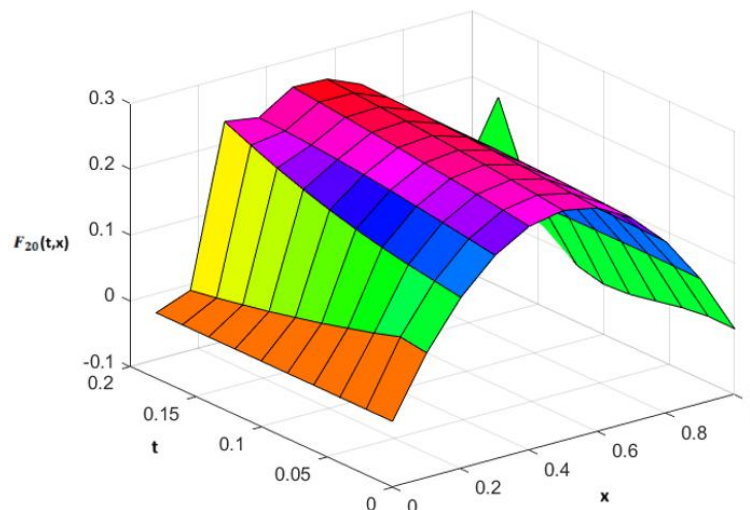


Fig. 12: F_{20} - the particle distribution moments.

$$F_{21}$$

$x \backslash t$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
t_1 0	0	0.0950	0.1800	0.2550	0.3200	0.3750	0.4200	0.4550	0.4800	0.4950	0.5000
t_2 0.02	0	0.0900	0.1756	0.2512	0.3167	0.3722	0.4177	0.4533	0.4788	0.4943	1.4263
t_3 0.04	0	0.0879	0.1712	0.2473	0.3134	0.3694	0.4155	0.4515	0.4775	0.4936	2.1956
t_4 0.06	0	0.0832	0.1668	0.2434	0.3100	0.3666	0.4132	0.4497	0.4762	0.4928	2.7944
t_5 0.08	0	0.0814	0.1624	0.2395	0.3066	0.3637	0.4108	0.4479	0.4749	0.4920	3.2095
t_6 0.1	0	0.0771	0.1580	0.2325	0.3032	0.3609	0.4085	0.4460	0.4736	0.4912	3.4282
t_7 0.12	0	0.0757	0.1537	0.2316	0.2998	0.3580	0.4061	0.4442	0.4723	0.4904	3.4381
t_8 0.14	0	0.0717	0.1494	0.2276	0.2963	0.3550	0.4037	0.4423	0.4709	0.4895	3.2279
t_9 0.16	0	0.0706	0.1451	0.2237	0.2928	0.3521	0.4012	0.4404	0.4695	0.4886	2.7873
t_{10} 0.18	0	0.0669	0.1408	0.2197	0.2893	0.3491	0.3988	0.4384	0.4681	0.4877	2.1086

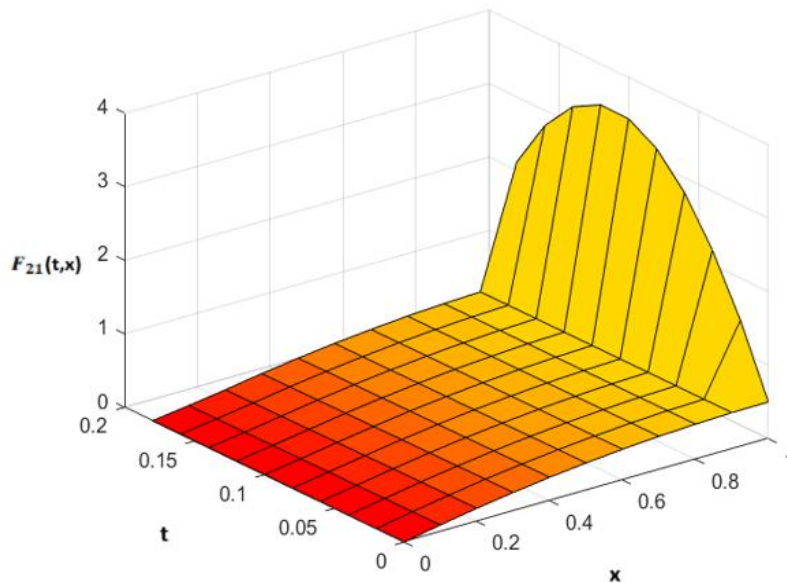


Fig. 13: F_{21} - the particle distribution moments.

4. Conclusion

We formulate initial boundary value problem for fifth approximation of one-dimensional non-linear non-stationary system of Boltzmann moment equations with macroscopic boundary conditions. We proved the theorem of existence and uniqueness of the local solution to the initial boundary value problem for one-dimensional system of Boltzmann moment equations with macroscopic boundary conditions in space of functions continuous in time and square-summable over the spatial variable, because the solution existence time depends on the norm of the initial vector function at the power minus one. Therefore, the smaller the norm of the initial vector function is, the longer the solution existence time of the initial boundary value problem for one-dimensional system of Boltzmann moment equations in fifth approximation and vice versa. In order to find an approximate solution to the initial boundary value problem for the system of Boltzmann moment equations, we used a numerical method. The differential problem is approximated by a finite-difference scheme, an algorithm and a program for numerical implementation on a computer are compiled. The moments of the particle distribution function are determined, which characterize atmospheric parameters, such as density, average speed and temperature, etc.

The obtained results are being used to establish the dependence of the solution to the initial boundary value problem for a non-stationary non-linear one-dimensional system of Boltzmann moment equations on the temperature of the reflecting wall. The approximate value of the particle distribution function corresponding to the fifth approximation of the moment equations has the form $F_5(t, x, v) = F_0(\alpha|v|) \sum_{2n+l=0}^5 F_{nl}(t, x) g_{nl}(\alpha v)$ $t > 0, x \in [0, 1]$, where the values of $F_{nl}(t, x)$ at $2n + l = 0, 1, \dots, 5$ are defined above (see tables of values $F_{00}, F_{01}, \dots, F_{21}$), and the values of the eigenfunctions $g_{nl}(\alpha v)$ are known.

The Maxwell boundary condition contains a term that depends on the wall temperature, which describes the particles reflected from the boundary according to the Maxwell distribution $F^+(t, x, v_1, v_2, v_3) = \beta F^-(t, x, v_1, v_2, -v_3) + (1 - \beta) \exp\left(-\frac{|v|^2}{2R\Theta}\right)$, where Θ is the temperature of the reflecting wall, $\beta \in (0, 1)$. By changing the values of the parameters Θ and β , we obtain different values of $F_5(t, x, v)$ depending on the wall temperature and the parameter β . For example, when $\beta = 1/2$, half of the particles $F_5^+(t, x, v_1, v_2, v_3)$ falling on the boundary are reflected from the boundary specularly, and the other half is reflected from the boundary diffusely following the Maxwell distribution. Thus,

approximate values of the particle distribution function will be determined depending on the temperature of the reflecting wall and the value of the parameter β . In general, the parameters Θ and β will depend on the coordinates and time. These studies concerning the establishment of the dependence of the solution to the initial boundary value problem for a non-stationary non-linear one-dimensional system of Boltzmann moment equations on the temperature of the reflecting wall and the parameter β will be carried out in subsequent works.

Conflict of Interest

There is no conflict of interest.

Supporting Information

Not applicable.

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